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GENERAL CONSIDERATION OF PROBLEMS IN COMPRESSIBLE FLOW USING THE HODOGRAPH METHOD

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SUMMARY

The purpose of the present report is to investigate the hodograph method as it is applied in general to the problem of compressible flow. First, the hodograph equations are given in various canonical forms which are convenient for obtaining solutions in the different flow regimes.

Since the coefficients of the canonical differential equations are implicit functions, exact solutions are difficult to find. Consequently different approximations are chosen so that some simpler differential equations capable of solution can be obtained. For most of the cases, fundamental or singular solutions are given or indicated.

The detailed development is concentrated on Chaplygin's second equation. The first-order approximation is well-known as the Tricomi equation. The second- and third-order approximations have a rather new approach. Both approximations follow the exact gas law closely in the neighborhood of the sonic velocity. The solutions are found to be Whittaker functions and the associated confluent hypergeometrical functions. Both approximations can be applied to the incompressible flow so that Chaplygin's procedure of borrowing the boundary conditions can be used if necessary. For the third-order approximation, the corresponding hypothetical gas law is derived and is found to differ very little from the exact gas law. The transformation relation between the hodograph plane and the physical plane is also given for the various solutions considered.

To make a comparison of the present approximate solution with the exact Chaplygin solutions, the flow through an aperture, studied by Chaplygin and Lighthill, is reexamined. There is some difference in the problem itself, as well as in the method of Chaplygin and Lighthill. First, the vessel with straight walls inclined at an arbitrary angle is considered rather than that with the wall at right angles. Second, no association of the boundary conditions with those for incompressible flow is made. The problem is treated directly as a boundary-value

problem. The result calculated with the Whittaker function checks well with that obtained by Chaplygin and Lighthill.

1 - INTRODUCTION

For the isentropic steady two-dimensional flow of nonviscous compressible fluids, the pair of differential equations in terms of the stream function ψ and the potential ϕ are nonlinear, and cannot be solved analytically except for a few special cases. Chaplygin (reference 1)[1] introduced the hodograph method in order to transform these nonlinear equations to linear ones, so that the available classical mathematical analysis and the principle of superposition can be applied. While there is a gain in the linearity of the equations, some new difficulties arise in the hodograph method. First of all, it is difficult, in general, to transfer the physical boundary conditions to the hodograph plane. Second, the flow in the hodograph plane usually consists of multiple-sheeted Riemann surfaces with a number of singularities where the analytic continuation of the series solutions becomes very complicated. Third, only one kind of a particular solution of the stream function - hypergeometrical functions and trigonometrical functions - has been obtained so far for the equations in the hodograph plane. By superposition, a series solution is achieved, but is difficult to apply even if the boundary conditions are known. Fourth, the transformation between the physical and hodograph planes becomes singular when the Jacobian determinate becomes zero or infinite. This is likely to happen when the supersonic flow is imbedded in a subsonic flow region. The existence of the so-called limiting line (reference 7) causes a breakdown of the entire flow.

In order to obtain some solutions in the case of subsonic flow, Chaplygin discovered an ingenious method of obtaining usable solutions for compressible flows by comparison with the corresponding series solutions of the incompressible-flow patterns as the limiting case in the hodograph plane. In the case of no circulation around the closed body, the derived series solutions involve a free constant to be chosen at pleasure. But fortunately each solution chosen corresponds to a reasonable flow pattern in the physical plane. As the free-stream Mach number increases, the body shape deviates from the image body in the incompressible flow. Thus, in general, the body shape cannot be preassigned and the compressible flow about it determined at a certain Mach number. This method has been carried further by Tsien and Kuo (references 2 and 3), Lighthill (reference 4), and Cherry (reference 5) on transonic flow up to the occurrence of the limiting line. Some of

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[1] A classified bibliography is given at the end of the paper.

their solutions apply to simple closed bodies with circulation. There are difficulties in the analytic continuation for an arbitrary flow pattern.

Although Chaplygin and others were pioneers in developing and adapting the hodograph method to gas dynamics in the early part of this century, results of the later Russian investigators were not available in the English language up to 1940. Since the second World War, the work of Russians has again assumed prominence in the field, namely, Frankl (reference 11), Falkovich (reference 34), and Christianovitch (reference 42). Along the line of formulation by Tricomi, Frankl's proof of the existence and the uniqueness of the solution of Chaplygin's equation in the transonic regime is interesting. Frankl and Falkovich obtained the solution for the channel flow and showed that the stream function is triple-valued in the hodograph plane of the axially symmetrical plane at sonic velocity. These results confirm what has been achieved by Lighthill (reference 4). Recently, Tomotika and Tamada (reference 33) have formulated some approximate nonlinear hodograph equations. A number of interesting particular solutions for the channel flow have been obtained. Of course, the solutions are not superimposable. Ehlers (reference 10) and Carrier, along the line of Christianovitch, have obtained the fundamental solutions of the Tricomi equation and the corresponding channel flow to the second order of approximation.

With the notion of the correspondence of the incompressible flow to the compressible, Bergman (reference 6) has developed an integration method for calculating the subsonic flow. For the supersonic case an extension of the Riemann method is also made by an iteration process (reference 41). Bers and Gelbart (reference 55) have similarly developed a line-integral operator to construct the so-called $\sum$-monogenic complex function which satisfies the hodograph equations or the general Cauchy-Riemann equations in equivalence. All are important contributions to the solution of the differential equation but offer rather difficult ways to obtain useful solutions for the flow of compressible fluids.

In the last few years, Guderley (reference 30) and Yoshihara (reference 36) have given a number of papers on transonic flows, particularly with the application of the transonic similarity law which was developed independently by Von Kármán (reference 29) and Guderley (reference 30). They have achieved very important approximate results in transonic flow.

In the subsonic case, if a linear approximation is used to replace the isentropic pressure-density relation, Kármán (reference 7) and

Tsien (reference 14), following another approach of Chaplygin, have obtained many useful results, particularly the well-known Kármán-Tsien formula (reference 7) for pressure corrections. Lin (reference 15) has discussed the conditions for obtaining closed contours in the physical plane, if circulation exists. Clauser (reference 24) recently applied this method to find the body shape with preassigned pressure distribution and this is definitely very useful in high-subsonic laminar-flow problems. Garrick and Kaplan (reference 20) have another interesting approach to pressure-correction formulas using Ringleb's solution (reference 8) of a simple source and vortex.

The present investigation, as the initial step of the research program of this challenging problem, is mainly interested in the following three aspects of the problem:

(a) Besides Chaplygin's differential equations (reference 1), can other useful forms of the differential equations be found systematically, particularly the canonical forms in the different flow regimes? Of particular interest are the fundamental or singular solutions which represent the types of singularities encountered in the hodograph plane. Is there any method of constructing such solutions as shown by Picard, Hadamard (reference 52), Hilbert, Riemann, and Tricomi (reference 53)?

(b) If such fundamental solutions are too difficult to construct or too complicated to apply to the flow problem, what other reasonable approximations, besides the Kármán-Tsien (reference 7) approach, can be made so that some useful results can be derived in the different flow regimes?

(c) It is well-known that the boundary conditions of the stream function ψ (not the potential ϕ) are well-defined in the hodograph for polygonal profiles with or without free streamlines up to transonic flow. As a reasonable preliminary approach, can the compressible flow around such a given body be found?

In brief, this paper contains a systematic list of useful forms of the differential equations. The canonical forms of the exact differential equations are given for subsonic, supersonic, and transonic regimes. Unfortunately, one of the coefficients of the equation is an implicit function of one independent variable. This makes it impossible to construct fundamental solutions which would be of value for practical application.

Approximations to the implicit function are chosen in such a way that the solutions can be found from classical mathematical analysis. In applying the approximations, three objectives are kept in mind: First, the differential equation must reduce to the Laplace equation as $M \longrightarrow 0$ or $M \longrightarrow M_\infty$ so that Chaplygin's procedure of utilizing the incompressible flow can be followed. Second, in the transonic range, the transonic similarity law of Kármán (reference 29) can be applied and the simplified boundary conditions can be used. Third, the singular solutions and the fundamental solutions in the subsonic and transonic regime can be found so that the flow can be determined directly if the boundary conditions in the hodograph plane are assigned. Or, alternately, the Riemann function in the supersonic regime can be found, so that an integral solution can be obtained when the initial value or the Cauchy data along a noncharacteristic line are specified. Last, the characteristics method can be used.

In the present paper, a number of approximations for the canonical forms in the subsonic flow are given. The zero-order approximation is actually the same as Von Kármán's approximation given in reference 7, equation (63), or reference 13, pages 186 to 188. Both differential equations of the first-order approximation can be reduced to the Laplace equation in polar coordinates. The singular and fundamental solutions have been given.

The second-order approximations to the differential equations can be reduced to Stratton's equation (reference 70) by means of the separation of variables. The particular solutions are indicated.

There are some better approximations which should hold for any subsonic Mach number. They are shown in section 3, "Solutions to Canonical Forms of Approximate Differential Equations in Subsonic, Supersonic, and Transonic Regimes." Similar approximations are obtained in the supersonic region. Of course, for each approximation, the corresponding pressure-density relation must be determined.

If the general class of singular solutions in the hodograph plane can be obtained, then it should be possible to solve a large number of problems by a suitable placement of the singularities in the hodograph plane to satisfy the desired boundary condition in much the same way that sources, vortices, doublets, and so forth are used in incompressible flow.

Some investigations on possible approximations to Chaplygin's second equation are made in section 4, "Different Approximations to Chaplygin's Differential Equation and Their Solutions." This second equation is more convenient to use in flow than the first equation. The first-order approximation to this differential equation has been shown by Frankl (reference 11) to be Tricomi's equation. Both the second- and third-order approximations are shown to be associated with Whittaker's equation (reference 61). The third approximation is sufficiently good that it should give fairly accurate results in all transonic-flow problems. The corresponding hypothetical gas law is shown in section 5, "Hypothetical Gas Law Corresponding to Approximations of Chaplygin's Second Equation," and deviates so little from the exact gas law in transonic range that the exact gas law may justifiably replace it. The transformation relation between the physical and hodograph planes is also shown for the third approximation.

The polygonal body, either closed or open, is known to give simple boundary conditions in the hodograph for the stream function ψ but not for the potential ϕ. The particular solutions obtained by the principle of separation of variables and the series solutions obtained by superposition cannot be applied directly without solving systems of an infinite number of simultaneous equations to determine the infinite number of coefficients in the series. In this class of bodies with one or more convex corners, the solution of the incompressible flow cannot be used because the velocity at such a corner is infinite according to the theory of incompressible flow. Physically, the flow passing such a body is always transonic in character, no matter how low the free-stream velocity is. As shown by Guderley (reference 31) and Busemann (reference 38) some shock always occurs at such corners. But with the simplified assumption of the transonic similarity law, the flow about this group of bodies should be obtainable. In section 3, "Solutions to Canonical Forms of Approximate Differential Equations in Subsonic, Supersonic, and Transonic Regimes," a more general simplified equation satisfying the transonic similarity law is given and the solutions are shown.

For an open body built with straight-line elements but with no convex corners the problem can be attacked with the Chaplygin technique or solved directly as a boundary-value problem. In this case, no free constants can be chosen, and the solution is uniquely determined.

Owing to the complicated nature of the asymptotic behavior of the Whittaker function with a very large parameter, and simultaneously with a very large value of the independent variable, the future work will devote a considerable amount of time to finding the asymptotic solutions corresponding physically to different flow regimes. This is an important step in solving flow problems with the Whittaker function. The Whittaker function converges very slowly for a large parameter, and is quite similar to the Chaplygin function in this respect.

In order to demonstrate the method, the flow through an aperture is reexamined in order to check with the results of Chaplygin and Lighthill. It is found that the third-order approximation agrees very well with their work. This is given in the last section of the paper.

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2 - CANONICAL AND OTHER FORMS OF DIFFERENTIAL EQUATIONS IN SUBSONIC, TRANSONIC, AND SUPERSONIC REGIMES

2.1 - General Transformation of Differential Equations in Hodograph Plane

The pair of Chaplygin's first-order simultaneous partial differential equations for the stream function ψ and the potential ϕ in the hodograph plane (equations (11.10) and (11.11) in reference 13) is

$$\frac{\partial\phi}{\partial q} = -\frac{\rho_0}{\rho}\frac{1 - M^2}{q}\frac{\partial\psi}{\partial\theta} \tag{1a}$$

$$\frac{\partial\phi}{\partial\theta} = \frac{\rho_0}{\rho} q \frac{\partial\psi}{\partial q} \tag{1b}$$

where ρ_0/ρ and $1 - M^2$ are given functions of the independent variable q.[2]

The above system can be transformed to many other forms which are more adaptable for analysis if a new independent variable Q is introduced to replace q, thus:

$$Q = Q(q) \tag{2}$$

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[2]A list of symbols is given in appendix A.

In terms of Q, equation (2) yields

$$\frac{\partial \phi}{\partial Q}\frac{dQ}{dq} = -\frac{\rho_o}{\rho}\frac{1 - M^2}{q}\frac{\partial \psi}{\partial \theta} \tag{3a}$$

$$\frac{\partial \phi}{\partial \theta} = \frac{\rho_o}{\rho} q \frac{dQ}{dq}\frac{\partial \psi}{\partial Q} \tag{3b}$$

Eliminating ϕ and then ψ from the above, a pair of second-order partial differential equations is obtained as follows:

$$\frac{\partial^2 \psi}{\partial Q^2} + \frac{1 - M^2}{q^2}\left(\frac{dQ}{dq}\right)^{-2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial \psi}{\partial Q}\frac{d}{dQ}\left[\log_e \left(\frac{\rho_o}{\rho} q \frac{dQ}{dq}\right)\right] = 0 \tag{4a}$$

$$\frac{\partial^2 \phi}{\partial Q^2} + \frac{1 - M^2}{q^2}\left(\frac{dQ}{dq}\right)^{-2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial \phi}{\partial Q}\frac{d}{dQ}\left\{\log_e \left[\frac{\rho_o}{\rho}\frac{1 - M^2}{q}\left(\frac{dQ}{dq}\right)^{-1}\right]\right\} = 0 \tag{4b}$$

Of course, if Q is chosen equal to q, the following pair of equations is obtained:

$$\frac{\partial^2 \psi}{\partial q^2} + \frac{1 - M^2}{q^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1 + M^2}{q}\frac{\partial \psi}{\partial q} = 0 \tag{4c}$$

$$\frac{\partial^2 \phi}{\partial q^2} + \frac{1 - M^2}{q^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{d}{dq}\left(\frac{\rho_o}{\rho}\frac{1 - M^2}{q}\right)\frac{\partial \phi}{\partial q} = 0 \tag{4d}$$

which have been used by Ringleb (reference 8). Now make some proper choice of the function $Q(q)$, so that either one or both of the above equations can be transformed to some simpler form.

2.2 - Chaplygin's Differential Equations - First Form

In Chaplygin's differential equations of the first form, he introduced the algebraic relation between Q and q as

$$Q(q) = \tau = \frac{\gamma - 1}{2} \frac{q^2}{\left(a_o{*}\right)^2} \tag{5}$$

where $a_o{*}$ is the stagnation sound velocity. It is obvious that the range of τ (τ has an important physical meaning. It is the ratio of the kinetic energy of the gas to the total energy or enthalpy) is from 0 to 1. Figure 1 shows the relation of τ, $\frac{q}{a_o{*}}$, and Mach number M. Equation (4a) transforms to his well-known equation

$$\frac{\partial}{\partial\tau}\left[\frac{2\tau}{(1 - \tau)^{\beta}} \frac{\partial\psi}{\partial\tau}\right] + \frac{1 - \mu_1\tau}{2\tau(1 - \tau)^{\beta+1}} \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{6a}$$

where $\beta = \frac{1}{\gamma - 1}$, $\mu_1 = \frac{\gamma + 1}{\gamma - 1}$, and the relations $\frac{\rho}{\rho_o} = (1 - \tau)^{\beta}$ and $1 - M^2 = \frac{1 - \mu_1\tau}{1 - \tau}$ are introduced. By means of the principle of the separation of variables, the particular solutions of this equation are combinations of hypergeometric and trigonometric functions. The differential equation in terms of the potential ϕ transforms to a much more complicated form

$$\frac{\partial}{\partial\tau}\left[\frac{2\tau(1 - \tau)^{\beta+1}}{1 - \mu_1\tau} \frac{\partial\phi}{\partial\tau}\right] + \frac{(1 - \tau)^{\beta}}{2\tau} \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{6b}$$

and its solution is difficult to obtain directly, but it can be obtained by integrating the differentials of ψ. Actually, there are four singularities of this equation - three regular ones at $\tau = 0$, $\tau = \frac{1}{\mu_1} = \frac{1}{2\beta + 1}$, and $\tau = 1$ and one irregular singularity at $\tau = \infty$. Therefore its solution is not so simple as the other equation.

2.3 - Chaplygin's Differential Equations - Second Form

To eliminate the last term in equation (4a), choose

$$Q(q) = \sigma \tag{7}$$

where σ is defined by (c_σ is a constant)

$$\frac{\rho_o}{\rho} q \frac{d\sigma}{dq} = c_\sigma \tag{8}$$

or

$$\sigma = c_\sigma \int \frac{\rho}{\rho_o} \frac{dq}{q} \tag{9}$$

Substituting into equation (4a), the last term drops out and

$$\frac{\partial^2\psi}{\partial\sigma^2} + \frac{K}{c_\sigma^{\ 2}} \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{10a}$$

where

$$K = \left(\frac{\rho_o}{\rho}\right)^2 \left(1 - M^2\right) = \frac{1 - \mu_1\tau}{(1 - \tau)^{\mu_1}} \tag{11}$$

In figure 2, K. is given as a function of τ. Of course, K should be expressed in terms of σ. Unfortunately, it is an implicit function of σ and the differential equation is impossible to solve exactly. Figure 3 shows the behavior of K as a function of σ for the case $c_\sigma = -1$ and $\gamma = 1.4$ where the upper limit of the integration is $q = a^*$, the sound velocity. It is of interest to observe the two asymptotes of the $K - \sigma$ curve: (a) $K \longrightarrow 1$ as $M \longrightarrow 0$ or $\sigma \longrightarrow \infty$ and (b) $K \longrightarrow -\infty$ as $\sigma \longrightarrow -0.2513$. The function σ can be integrated as follows:

$$\sigma = \int_q^{a^*} \frac{\rho}{\rho_o} \frac{dq}{q}$$

$$= -0.2513 + \tanh^{-1}(1 - \tau)^{1/2} - (1 - \tau)^{1/2}\left[1 + \frac{1 - \tau}{3} + \frac{(1 - \tau)^2}{5}\right] \tag{12}$$

where τ is given in equation (5) and $\gamma = 1.4$ is explicitly introduced. In figure 4, σ is given as a function of τ. If equation (9) is substituted into equation (4b),

$$\frac{\partial^2 \phi}{\partial \sigma^2} + K \frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial \phi}{\partial \sigma} \frac{d}{d\sigma}\left(\log_e K\right) = 0 \tag{10b}$$

which is rather more complicated.

2.4 - Another Form of Chaplygin's Second Differential Equation

To eliminate the last term in equation (4b), introduce

$$Q(q) = \nu \tag{13}$$

where ν is defined by (c_ν is a constant)

$$\frac{\rho}{\rho_o} \frac{q}{1 - M^2}\left(\frac{d\nu}{dq}\right) = c_\nu \tag{14}$$

or

$$\nu = -c_\nu \int_q^{a^*} \frac{\rho_o}{\rho} \frac{1 - M^2}{q} dq \tag{15}$$

Substituting in equation (4b), the last term drops out and

$$K \frac{\partial^2 \phi}{\partial \nu^2} + \frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{16}$$

where K is the same as defined in equation (11) and $c_\nu = -1$ is chosen. Of course K should be expressed in terms of ν, if it were not an implicit function of ν. No detailed investigation will be given in this report.

Similarly it can be shown that

$$K \frac{\partial^2\psi}{\partial\nu^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\nu}\frac{d}{d\nu}\left(\log_e K\right) = 0 \tag{16a}$$

It should be noted that equations (10) or (16) are simpler for only one of the pair, at the sacrifice of the other.

2.5 - Canonical Forms in Subsonic Flow

It is well-known that the fundamental solution may be found when the differential equation is reduced to the canonical form. Therefore, the canonical form is worth while to obtain. If the velocity is subsonic everywhere in the domain, the canonical forms of the differential equations can be found in equation (4) in the subsonic or elliptic range. Take

$$Q(q) = \omega \tag{17}$$

such that, for $M < 1$,

$$\frac{1 - M^2}{q^2}\left(\frac{d\omega}{dq}\right)^{-2} = 1$$

or

$$d\omega = -\frac{\left(1 - M^2\right)^{1/2}}{q}\,dq \tag{18a}$$

$\left(\text{Actually } d\omega = \pm\left(1 - M^2\right)^{1/2}\frac{dq}{q}\right.$. Here the minus sign is chosen so that a^* can be imposed as an upper limit.$\left.\right)$ After integration,

$$\omega = \int_{q}^{a^*} \frac{\left(1 - M^2\right)^{1/2}}{q} \, dq \tag{18b}$$

Figure 5 shows ω as a function of τ, M, and σ. In terms of M or τ,

$$\omega = \mu_1{}^{1/2} \tanh^{-1}\left(\frac{1 - M^2}{\mu_1}\right)^{1/2} - \tanh^{-1}\left(1 - M^2\right)^{1/2}$$

$$= \mu_1{}^{1/2} \tanh^{-1}\left[\frac{1 - \mu_1\tau}{\mu_1(1 - \tau)}\right]^{1/2} - \tanh^{-1}\left(\frac{1 - \mu_1\tau}{1 - \tau}\right)^{1/2} \tag{18c}$$

Equations (4a) and (4b) become

$$\frac{\partial^2\psi}{\partial\omega^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\omega}\frac{d}{d\omega}\left(\log_e K^{1/2}\right) = 0 \tag{19a}$$

$$\frac{\partial^2\phi}{\partial\omega^2} + \frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\omega}\frac{d}{d\omega}\left(\log_e K^{1/2}\right) = 0 \tag{19b}$$

If $\psi = K^{-1/4}\psi*$ and $\phi = K^{1/4}\phi*$ are introduced, the above equations can be written as

$$\frac{\partial^2\psi*}{\partial\omega^2} + \frac{\partial^2\psi*}{\partial\theta^2} + \left[\frac{3}{16}\left(\frac{K'}{K}\right)^2 - \frac{1}{4}\frac{K''}{K}\right]\psi* = 0 \tag{19c}$$

$$\frac{\partial^2\phi*}{\partial\omega^2} + \frac{\partial^2\phi*}{\partial\theta^2} + \left[-\frac{1}{16}\left(\frac{K'}{K}\right)^2 + \frac{1}{4}\frac{K''}{K}\right]\phi* = 0 \tag{19d}$$

which are the canonical forms of the differential equations in subsonic flow and were first pointed out by Bergman (reference 6).

$$\frac{1}{2}\frac{K'}{K} = \frac{d\left(\log_e K^{1/2}\right)}{d\omega}$$

$$= \frac{-\left(\mu_1 - 1\right)\mu_1\tau^2}{\left(1 - \mu_1\tau\right)^{3/2}\left(1 - \tau\right)^{1/2}}$$

$$= -\frac{\mu_1}{\mu_1 - 1}\frac{M^4}{\left(1 - M^2\right)^{3/2}} \tag{20}$$

Figure 6 shows $\dfrac{d\left(\log_e K^{1/2}\right)}{d\omega}$ as a function of ω.

Unfortunately the coefficient of the first derivative is an implicit function of ω and also is singular at $M = 1$. This prevents the use of Hilbert's and Hadamard's approach (reference 52) to formulate the fundamental solution. Bergman (reference 6), Bers (reference 17), and Gelbart (reference 27) in this country and Eichler (reference 58) in Germany have given the integral solution of equation (19) but the process is very complicated. The present paper will give some solutions of the above differential equations (19) under certain approximations.

2.6 - Canonical Forms in Supersonic Range

If the velocity q in the flow is entirely supersonic, take

$$Q(q) = \Omega \tag{21}$$

where Ω is defined by

$$\frac{M^2 - 1}{q^2}\left(\frac{d\Omega}{dq}\right)^2 = 1 \tag{22a}$$

or, if the positive sign is chosen,

$$d\Omega = \frac{\left(M^2 - 1\right)^{1/2}}{q}\,dq$$

After integration,

$$\Omega = \int_{a*}^{q} \frac{\left(M^2 - 1\right)^{1/2}}{q} dq \tag{22b}$$

where Ω can be expressed in terms of M or τ, each in two ways,

$$\Omega = {\mu_1}^{1/2} \tan^{-1}\left(\frac{M^2 - 1}{\mu_1}\right)^{1/2} - \tan^{-1}\left(M^2 - 1\right)^{1/2}$$

$$= {\mu_1}^{1/2} \tan^{-1}\left[\frac{\mu_1\tau - 1}{\mu_1(1 - \tau)}\right]^{1/2} - \tan^{-1}\left(\frac{\mu_1\tau - 1}{1 - \tau}\right)^{1/2} \tag{22c}$$

or

$$\Omega = \mu^{1/2} \cos^{-1}\left[\frac{\mu_1}{M^2 + \left(\mu_1 - 1\right)}\right]^{1/2} - \cos^{-1} \frac{1}{M}.$$

$$= {\mu_1}^{1/2} \cos^{-1}\left[\frac{\mu_1(1 - \tau)}{\mu_1 - 1}\right]^{1/2} - \cos^{-1}\left[\frac{1 - \tau}{\left(\mu_1 - 1\right)\tau}\right]^{1/2} \tag{22d}$$

Figure 5 also shows Ω as a function of τ. Actually Ω is a distorted velocity magnitude such that $\Omega = \pm\theta$ become two families of simple straight characteristics inclined at $\pm 45^{\circ}$. (Refer to p. 215 of reference 13.) Equation (4) becomes

$$\frac{\partial^2\psi}{\partial\Omega^2} - \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\Omega} \frac{d}{d\Omega}\left[\log_e (-K)^{1/2}\right] = 0 \tag{23a}$$

$$\frac{\partial^2\phi}{\partial\Omega^2} - \frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\Omega} \frac{d}{d\Omega}\left[\log_e (-K)^{1/2}\right] = 0 \tag{23b}$$

In the hodograph plane, the characteristics are the same as those of the simple wave equation and are invariant to the behavior of $K(\Omega)$. If $\psi = (-K)^{-1/4}\psi^*$ and $\phi = (-K)^{1/4}\phi^*$ are introduced, the following equations are obtained:

$$\frac{\partial^2\psi^*}{\partial\Omega^2} - \frac{\partial^2\psi^*}{\partial\theta^2} + \left[\frac{3}{16}\left(\frac{K'}{K}\right)^2 - \frac{1}{4}\frac{K''}{K}\right]\psi^* = 0 \tag{23c}$$

$$\frac{\partial^2\phi^*}{\partial\Omega^2} - \frac{\partial^2\phi^*}{\partial\theta^2} + \left[-\frac{1}{16}\left(\frac{K'}{K}\right)^2 + \frac{1}{4}\frac{K''}{K}\right]\phi^* = 0 \tag{23d}$$

$$\begin{aligned}\frac{d\left[\log_e (-K)^{1/2}\right]}{d\Omega} &= \frac{\left(\mu_1 - 1\right)\mu_1\tau^2}{\left(\mu_1\tau - 1\right)^{3/2}(1 - \tau)^{1/2}} \\ &= \frac{\mu_1}{\mu_1 - 1}\frac{M^4}{\left(M^2 - 1\right)^{3/2}}\end{aligned} \tag{24}$$

Figure 7 gives $\dfrac{d\left[\log_e (-K)^{1/2}\right]}{d\Omega}$ as a function of Ω. The Riemann function (reference 52) always exists uniquely and the solution of both equations can be constructed theoretically for a region bounded by characteristics if the Cauchy data are given along a line that is not a characteristic. Owing to the implicit nature of the coefficient $\dfrac{d}{d\Omega}\left[\log_e (-K)^{1/2}\right]$ as a function of Ω, the Riemann function and the solution will be too complicated to construct. Besides for the supersonic flow the Cauchy data in the transformed hodograph plane are not completely known in general. Therefore the Riemann function is not of much use in the hodograph method.

The present paper will also show some solutions to the differential equations under certain approximations. It should be noted that Ω and ω are related by $\Omega = i\omega$ if it is desired to extend the definition of ω to the supersonic side. Such an extension seems very obscure as yet in its meaning. Fortunately $(d\Omega)^2$ always occurs in differentiation, and the derivative of the logarithmic term does not depend on such an extension. Therefore equations (23a) and (23b) are single-valued.

There is another interesting feature of the canonical form. In the hyperbolic region, the characteristics are the same as those of the simple wave equation while in the elliptic region the imaginary characteristics are the same as those of the Laplace equation. They are invariant to physical conditions. The physical law influences only the first-derivative term.

2.7 - Canonical Forms in Transonic Domain

Now choose $Q(q) = \epsilon$ where ϵ is defined by

$$\frac{1 - M^2}{q^2}\left(\frac{d\epsilon}{dq}\right)^{-2} = \pm f(\epsilon) \tag{25}$$

and where $f(\epsilon)$ is an arbitrary function. (The upper sign is to be used for subsonic flow and the lower sign for supersonic flow.) Then,

$$\int \left[f(\epsilon)\right]^{1/2} d\epsilon = \int \frac{\left[\pm\left(1 - M^2\right)\right]^{1/2}}{q} dq \tag{26}$$

For $M < 1$ choose

$$f(\epsilon')\left(\frac{d\epsilon'}{dq}\right)^2 = \frac{1 - M^2}{q^2} \tag{25a}$$

Equations (4a) and (4b) can be written as

$$\frac{\partial^2\psi}{\partial\epsilon'^2} + f(\epsilon')\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\epsilon'}\frac{d}{d\epsilon'}\left\{\log_e\left[\frac{K}{f(\epsilon')}\right]^{1/2}\right\} = 0 \tag{27a}$$

$$\frac{\partial^2\phi}{\partial\epsilon'^2} + f(\epsilon')\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\epsilon'}\frac{d}{d\epsilon'}\left\{\log_e\left[Kf(\epsilon')\right]^{1/2}\right\} = 0 \tag{27b}$$

where ϵ' is used for ϵ in the subsonic region. For $M > 1$ choose

$$f(\epsilon'')\left(\frac{d\epsilon''}{dq}\right)^2 = \frac{M^2 - 1}{q^2} \tag{25b}$$

$$\frac{\partial^2\psi}{(\partial\epsilon'')^2} - f(\epsilon'')\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\epsilon''}\frac{d}{d\epsilon''}\left\{\log_e\left[\frac{(-K)}{f(\epsilon'')}\right]^{1/2}\right\} = 0 \tag{28a}$$

$$\frac{\partial^2\phi}{(\partial\epsilon'')^2} - f(\epsilon'')\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\epsilon''}\frac{d}{d\epsilon''}\left\{\log_e\left[(-K)f(\epsilon'')\right]^{1/2}\right\} = 0 \tag{28b}$$

where ϵ'' is used for ϵ in the supersonic region.

It is interesting to note that the above two sets of equations can be combined into a single set if $f(\epsilon)$ is an odd function, that is,

$$f(\epsilon)\left(\frac{d\epsilon}{dq}\right)^2 = \frac{M^2 - 1}{q^2} \tag{25c}$$

$$\frac{\partial^2\psi}{\partial\epsilon^2} - f(\epsilon)\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\epsilon}\frac{d}{d\epsilon}\left\{\log_e\left[\frac{(-K)}{f(\epsilon)}\right]^{1/2}\right\} = 0 \tag{29a}$$

$$\frac{\partial^2\phi}{\partial\epsilon^2} - f(\epsilon)\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\epsilon}\frac{d}{d\epsilon}\left\{\log_e\left[(-K)f(\epsilon)\right]^{1/2}\right\} = 0 \tag{29b}$$

which are valid for both $M < 1$ and $M > 1$. It should be noted that $\epsilon > 0$ for $M > 1$ and $\epsilon < 0$ for $M < 1$.

The canonical form of the mixed differential equation is explained as follows. If, following Tricomi (reference 52) $f(\epsilon)$ is set equal to ϵ and $-K = \left(\frac{\rho_0}{\rho}\right)^2\left(M^2 - 1\right)$ is introduced, equation (29) reduces to

$$\frac{\partial^2 \psi}{\partial \epsilon^2} - \epsilon \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{2} \frac{\partial \psi}{\partial \epsilon} \frac{d}{d\epsilon}\left[\log_e \left(\frac{-K}{\epsilon}\right)\right] = 0 \tag{30a}$$

$$\frac{\partial^2 \phi}{\partial \epsilon^2} - \epsilon \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{2} \frac{\partial \phi}{\partial \epsilon} \frac{d}{d\epsilon}\left[\log_e (-\epsilon K)\right] = 0 \tag{30b}$$

which are valid for the mixed region. They can be obtained directly by setting $\epsilon^3 = \left(\frac{3}{2}\,\Omega\right)^2$ in equation (23) if $M > 1$ and $-\epsilon^3 = \left(\frac{3}{2}\,\omega\right)^2$ in equation (19) if $M < 1$. These may be called the canonical forms of the differential equations in the transonic region. The solution of the exact differential equation is obviously difficult. Some singular solutions of the above equation under approximations can be obtained as will be shown later.

It is well-known that the characteristics are fixed for all linear hyperbolic differential equations. But there is an interesting feature in the canonical form such as equations (30a) and (30b), because the characteristics are fixed and invariant to the function $K(\epsilon)$ or to the physical problem from which the differential equation is derived. The characteristic equations are

$$\left(\theta - \theta_o\right)^2 - \frac{4}{9}\epsilon^3 = 0$$

which represent two families of characteristics of cubic parabolas with cusps at their points of intersection with the $\epsilon = 0$ axis.

Figure 8 shows ϵ as a function of ω and Ω. Figure 9 shows $\epsilon \frac{d}{d\epsilon}\left[\log_e \left(\frac{-K}{\epsilon}\right)^{1/2}\right]$ as a function of ϵ.

3 - SOLUTIONS TO CANONICAL FORMS OF APPROXIMATE DIFFERENTIAL EQUATIONS IN SUBSONIC, SUPERSONIC, AND TRANSONIC REGIMES

In the last section, the general derivatives were shown of the canonical forms of the differential equations in the different regimes. At the same time, it is found that the coefficient of the first-derivative term in any one of the canonical forms is always an implicit function of one independent variable. The exact solutions are very difficult to obtain. It seems justifiable to seek some differential equations with an approximate coefficient for the first-derivative term so that solutions are relatively easy to formulate. Of course, the validity of the solutions thus obtained must be examined and the limits of the range of application must be defined. In this section, the solutions of the approximate differential equations will be given and their applications will be discussed.

3.1 - Subsonic Canonical Forms - Equations (19a) and (19b)

In the subsonic case three approximations to the canonical form of the differential equation will be given. Figure 10 shows the exact curve of $K^{1/2} = \frac{\rho_0}{\rho}\left(1 - M^2\right)^{1/2}$ against ω.

Zero-order approximation of Von Kármán.- In reference 7, Von Kármán suggests taking

$$K^{1/2} = \frac{\rho_0}{\rho}\left(1 - M^2\right)^{1/2} \approx \frac{\rho_0}{\rho_\infty}\left(1 - M_\infty^2\right)^{1/2} \tag{31}$$

where ρ_∞ and M_∞ are the density and Mach number at the free stream. In figure 10, the horizontal dotted line shows the approximation $K^{1/2}_{\omega=\omega_\infty}$. It is apparent that the approximation becomes better as $\omega \longrightarrow 0$ or $M \longrightarrow 0$. With this approximation, the coefficient of the first derivative becomes zero in equation (19) and so

$$\frac{\partial^2\psi}{\partial\omega^2} + \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{32a}$$

$$\frac{\partial^2\phi}{\partial\omega^2} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{32b}$$

which are Laplace equations. These equations are invariant under translation and rotation. Therefore, ω can be set equal to 0 at sonic velocity a^* and so defined as

$$\omega = \int_q^{a^*} \frac{\left(1 - M^2\right)^{1/2}}{q'} dq' \tag{33}$$

Of course, Chaplygin's procedure of utilizing the incompressible-flow pattern to find the corresponding compressible flow can be followed. There is another possible approach. If the types of singularities and their locations in the flow of the hodograph plane are known and if, in addition, the boundary conditions of the stream function are known under reasonable assumption or under assignment, the solution of the flow should be obtained uniquely by superposition of the fundamental solutions corresponding to the types and location of the singularities. After the solution in the hodograph plane is obtained, the corresponding flow in the physical plane should be checked to determine if it is possible or not. To serve such a purpose, the fundamental solutions of ψ and ϕ due to a source at ω_o and θ_o are given as an example.

$$\phi\left(\omega,\theta;\ \omega_o,\theta_o\right) = -\log_e \left[\left(\omega - \omega_o\right)^2 + \left(\theta - \theta_o\right)^2\right]^{1/2} \tag{34a}$$

$$\psi\left(\omega,\theta;\ \omega_o,\theta_o\right) = -\tan^{-1} \frac{\omega - \omega_o}{\theta - \theta_o} \tag{34b}$$

Of course, with the simple source, there can be built up the potential and the stream function of higher-order singularities such as doublets and quadrupoles. Actually, it is more convenient to treat the problems with functions of complex variables, because ψ and ϕ are harmonic functions.

First-order approximation of the canonical form.- It has been known for some time that the above approximation is not very good at high-subsonic velocity. A higher order of approximation can be made as follows. Instead of taking the approximation as given in equation (31), the approximation can be made accurate to the slope of the curve $K^{1/2}$ against ω at the free-stream condition, that is,

$$K^{1/2} \approx a + b\omega \tag{35a}$$

where

$$a = \left(K^{1/2} - \omega \frac{dK^{1/2}}{d\omega}\right)_{\omega=\omega_\infty} \tag{35b}$$

$$b = \left(\frac{dK^{1/2}}{d\omega}\right)_{\omega=\omega_\infty} \tag{35c}$$

if $\omega = \omega_\infty$ is chosen as the free-stream condition. This approximation is shown in figure 10. Then equation (19) yields:

$$\frac{\partial^2\psi}{\partial\omega^2} + \frac{\partial^2\psi}{\partial\theta^2} - \frac{b}{a + b\omega}\frac{\partial\psi}{\partial\omega} = 0 \tag{36a}$$

$$\frac{\partial^2\phi}{\partial\omega^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{b}{a + b\omega}\frac{\partial\phi}{\partial\omega} = 0 \tag{36b}$$

Introducing $\overline{\omega} = \omega + \frac{a}{b}$ into the above equations,

$$\frac{\partial^2\psi}{\partial\overline{\omega}^2} + \frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial\overline{\omega}} = 0 \tag{37a}$$

$$\frac{\partial^2\phi}{\partial\overline{\omega}^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\overline{\omega}}\frac{\partial\phi}{\partial\overline{\omega}} = 0 \tag{37b}$$

These are identical to the equations of Stokes and Beltrami for axially symmetrical flow. The particular solutions of interest are the well-known Bessel and trigonometric functions, and are not given here.

These differential equations are invariant under translation in θ. From Lamb's "Hydrodynamics," the singular solution corresponding to a source at $\omega_0 = -\frac{a}{b}$ or $\overline{\omega}_0 = 0$, θ_0 is

$$\phi\left(\omega + \frac{a}{b},\theta;\ -\frac{a}{b},\theta_o\right) = \frac{1}{\left[\left(\omega + \frac{2a}{b}\right)^2 + \left(\theta - \theta_o\right)^2\right]^{1/2}} \tag{38}$$

Recently Weinstein (reference 56) has shown that, for the axially symmetrical potential, the fundamental solution for a source at $\left(\omega_o + \frac{a}{b},\theta_o\right)$ is

$$\phi\left(\omega + \frac{a}{b},\theta;\ \omega_o + \frac{a}{b},\theta_o\right) = \frac{-\log_e\left[\left(\omega - \omega_o\right)^2 + \left(\theta - \theta_o\right)^2\right]^{1/2}}{\pi\left(\omega_o + \frac{a}{b}\right)^{1/2}\left(\omega + \frac{a}{b}\right)^{1/2}} + R\left(\omega + \frac{a}{b},\theta\right) \tag{39}$$

where the function $R\left(\omega + \frac{a}{b},\theta\right)$ is regular at the point $\left(\omega_o + \frac{a}{b},\theta_o\right)$.

The corresponding singular solution to the stream function is

$$\psi\left(\omega + \frac{a}{b},\theta;\ -\frac{a}{b},\theta_o\right) = \frac{\theta - \theta_o}{\left[\left(\omega + \frac{2a}{b}\right)^2 + \left(\theta - \theta_o\right)^2\right]^{1/2}} \tag{40}$$

The fundamental solution of the stream function is not given here, because it is long and involved. The singular solution for more complicated sources can be built up easily. Therefore, if the boundary conditions and the locations and types of sources are given in the hodograph plane, the flow in the hodograph plane can be found with the classical technique of boundary-value problems. If the boundary conditions are not known, the series solution of the incompressible flow may be considered and Chaplygin's procedure followed.

Second-order approximations.- Choose the following approximation:

$$K^{1/2} \approx a + b\omega + c\omega^2 \tag{41a}$$

where

$$a = \left[K^{1/2} - \omega\left(\frac{dK^{1/2}}{d\omega}\right) + \frac{1}{2!}\,\omega^2\left(\frac{d^2K^{1/2}}{d\omega^2}\right)\right]_{\omega=\omega_\infty} \tag{41b}$$

$$b = \left(\frac{dK^{1/2}}{d\omega} - \omega \frac{d^2K^{1/2}}{d\omega^2}\right)_{\omega=\omega_\infty} \tag{41c}$$

$$c = \frac{1}{2!}\left(\frac{d^2K^{1/2}}{d\omega^2}\right)_{\omega=\omega_\infty} \tag{41d}$$

This approximation is also shown in figure 10. Then equation (19) yields

$$\frac{\partial^2\psi}{\partial\omega^2} + \frac{\partial^2\psi}{\partial\theta^2} - \frac{b + 2c\omega}{a + b\omega + c\omega^2}\frac{\partial\psi}{\partial\omega} = 0 \tag{42a}$$

$$\frac{\partial^2\phi}{\partial\omega^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{b + 2c\omega}{a + b\omega + c\omega^2}\frac{\partial\phi}{\partial\omega} = 0 \tag{42b}$$

These equations can be written in another form by introducing $\omega' = \frac{2c}{\left(b^2 - 4ac\right)^{1/2}}\left(\omega - \frac{b}{2c}\right)$ $\left(\text{if} \quad b^2 > 4ac\right)$.

$$\frac{\partial^2\psi}{\partial\omega'^2} + \frac{\partial^2\psi}{\partial\theta^2} - \left(\frac{1}{\omega' - 1} + \frac{1}{\omega' + 1}\right)\frac{\partial\psi}{\partial\omega'} = 0 \tag{43a}$$

$$\frac{\partial^2\phi}{\partial\omega'^2} + \frac{\partial^2\phi}{\partial\theta^2} + \left(\frac{1}{\omega' - 1} + \frac{1}{\omega' + 1}\right)\frac{\partial\phi}{\partial\omega'} = 0 \tag{43b}$$

The particular solution can be obtained by the principle of separation of variables, if it is assumed that

$$\psi(\omega',\theta) = \psi_1^{(\nu)}(\omega')\psi_2^{(\nu)}(\theta) = \left.\begin{matrix}\sin \nu\theta \\ \cos \nu\theta\end{matrix}\right\} \psi_1^{(\nu)}(\omega') \tag{44a}$$

$$\phi(\omega',\theta) = \phi_1^{(\nu)}(\omega')\phi_2^{(\nu)}(\theta) = \left.\begin{matrix}\sin \nu\theta \\ \cos \nu\theta\end{matrix}\right\} \phi_1^{(\nu)}(\omega') \tag{44b}$$

where ψ_1 and ϕ_1 satisfy the ordinary equations

$$\frac{d^2\psi_1}{(d\omega')^2} - \frac{2\omega'}{(\omega')^2 - 1}\frac{d\psi_1}{d\omega'} - \nu^2\psi_1 = 0 \tag{45a}$$

$$\frac{d^2\phi_1}{(d\omega')^2} + \frac{2\omega'}{(\omega')^2 - 1}\frac{d\phi_1}{d\omega'} - \nu^2\phi_1 = 0 \tag{45b}$$

which are special cases of Stratton's equation as shown in reference 70. The particular series solution can be obtained but it is too long to give here.

Comments on above three approximations.- If the above three approximations are plotted in comparison with $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ against ω in figure 6, they are not satisfactory. The zero-order approximation maintains a zero value, although the true value of $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ at ω_∞ may not be zero. The first-order approximation becomes roughly the zero-order approximation to $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ at ω_∞, while the second-order approximation becomes roughly the first-order one. It is apparent that the above procedure can be repeated to find all the approximations to the curve in figure 11, but this kind of approximation is not shown here. Some other approximations are given to take care of the asymptotic behavior of $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ as $\omega \longrightarrow 0$ and $\omega \longrightarrow \infty$.

Approximations to singular behavior of $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$.- Figure 11 shows the curve of $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ against ω. Some simple calculations will show that $\lim_{\omega \to 0} \omega \frac{d}{d\omega}\left(\log_e K^{1/2}\right) = -\frac{1}{3}$ and $\frac{d}{d\omega}\left[\log_e \frac{\rho_o}{\rho}\left(1 - M^2\right)^{1/2}\right] \longrightarrow 0$ as $\omega \longrightarrow \infty$. Therefore, a simple, but good, approximation is to assume

$$\frac{d}{d\omega}\left(\log_e K^{1/2}\right) = \frac{K'}{2K} \approx \frac{a}{\omega} \tag{46a}$$

The constant a can be chosen in two ways:

(1) Choose $a = -\frac{1}{3}$ if the exact behavior at sonic velocity is desired. This is shown in figure 11.

(2) Choose a such that $\frac{a}{\omega_\infty} = \left(\frac{K'}{2K}\right)_{\omega=\omega_\infty}$, if the flow velocity in the neighborhood of free-stream velocity is desired. If ω_∞ is infinitely large, it automatically reduces to Kármán's approximation. This case is also shown in figure 11.

This approximation is important because the fundamental and singular solutions can be obtained similarly to equations (38) and (39) except the one-half power is replaced by the $\frac{a}{2}$ power.

The next approximation along this line is to assume

$$\frac{d}{d\omega}\left(\log_e K^{1/2}\right) \approx \frac{a}{\omega(1 + b\omega)} \tag{46b}$$

where a and b are two free constants to be chosen.

A few choices can be made:

(1) Take $a = -\frac{1}{3}$ and b to be such that $\frac{K'}{2K} = \frac{-1/3}{\omega_\infty\left(1 + b\omega_\infty\right)}$ if the interest is in the sonic and free-stream velocities.

(2) Take a and b such that $\frac{a}{\omega(1 + b\omega)} = \left(\frac{K'}{2K}\right)_{\omega=\omega_\infty}$ and $\left(\frac{K'}{2K}\right)_{\omega=0.5\omega_\infty}$ are satisfied, for example. This case, shown in the graph (fig. 11) checks well with the exact curve. The differential equations are simplified to

$$\frac{\partial^2\psi}{\partial\omega^2} + \frac{\partial^2\psi}{\partial\theta^2} - \frac{a}{\omega(1 + b\omega)}\frac{\partial\psi}{\partial\omega} = 0 \tag{47a}$$

$$\frac{\partial^2\phi}{\partial\omega^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{a}{\omega(1 + b\omega)}\frac{\partial\phi}{\partial\omega} = 0 \tag{47b}$$

Similarly, it can further be assumed that

$$\frac{K'}{2K} \approx \frac{a(1 + b\omega)}{\omega(1 + c\omega)} \tag{46c}$$

where a, b, and c are free constants. One case is shown in the graph (fig. 11) with a, b, and c determined from the values of $\frac{K'}{2K}$ at $0.5\omega_\infty$, ω_∞, and $2\omega_\infty$. It checks very well with the exact curve.

3.2 - Supersonic Canonical Forms - Equations (23a) and (23b)

Zero-order approximation.- Following the approach of Von Kármán in the subsonic case, there can be chosen in the supersonic case the approximation

$$(-K)^{1/2} = \frac{\rho_o}{\rho}\left(M^2 - 1\right)^{1/2} \approx \frac{\rho_o}{\rho_\infty}\left(M_\infty^2 - 1\right)^{1/2} \tag{48a}$$

and

$$\Omega = \int_{a^*}^{q} \frac{\left(M^2 - 1\right)^{1/2}}{q'} \, dq' \tag{48b}$$

where q is always supersonic in the domain of the flow pattern. Then, the coefficient of the first derivative in equation (23) is zero, and

$$\frac{\partial^2\psi}{\partial\Omega^2} - \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{49a}$$

$$\frac{\partial^2\phi}{\partial\Omega^2} - \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{49b}$$

which has the solutions

$$\psi(\Omega,\theta) = F_\psi(\Omega + \theta) + G_\psi(\Omega - \theta)$$

$$\phi(\Omega,\theta) = F_\phi(\Omega + \theta) + G_\phi(\Omega - \theta)$$

which are well-known classical wave functions. The approximation is a straight line parallel to the Ω-axis as shown in figure 12 with $\Omega = \Omega_\infty$.

First-order approximation.- Along the same approach as the subsonic case, it is easy to show

$$\frac{\partial^2\psi}{\partial\Omega^2} - \frac{\partial^2\psi}{\partial\theta^2} - \frac{b'}{a' + b'\Omega}\frac{\partial\psi}{\partial\Omega} = 0 \tag{50a}$$

$$\frac{\partial^2\phi}{\partial\Omega^2} - \frac{\partial^2\phi}{\partial\theta^2} + \frac{b'}{a' + b'\Omega}\frac{\partial\phi}{\partial\Omega} = 0 \tag{50b}$$

where

$$a' = \left[(-K)^{1/2} - \Omega \frac{d(-K)^{1/2}}{d\Omega}\right]_{\Omega=\Omega_\infty}$$

$$b' = \left[\frac{d(-K)^{1/2}}{d\Omega}\right]_{\Omega=\Omega_\infty}$$

Introducing $\overline{\Omega} = \Omega + \frac{a'}{b'}$ into the above equations,

$$\frac{\partial^2\psi}{\partial\overline{\Omega}^2} = \frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{\overline{\Omega}}\frac{\partial\psi}{\partial\overline{\Omega}} = 0 \tag{51a}$$

$$\frac{\partial^2\phi}{\partial\overline{\Omega}^2} - \frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\overline{\Omega}}\frac{\partial\phi}{\partial\overline{\Omega}} = 0 \tag{51b}$$

If the following characteristic coordinates are introduced,

$$\lambda = \overline{\Omega} + \theta \tag{52a}$$

$$\mu = -\overline{\Omega} + \theta \tag{52b}$$

and then,

$$\frac{\partial^2\psi}{\partial\lambda\,\partial\mu} + \frac{1}{2(\lambda - \mu)}\left(\frac{\partial\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\mu}\right) = 0 \tag{53a}$$

$$\frac{\partial^2\phi}{\partial\lambda\,\partial\mu} - \frac{1}{2(\lambda - \mu)}\left(\frac{\partial\phi}{\partial\lambda} - \frac{\partial\phi}{\partial\mu}\right) = 0 \tag{53b}$$

which are Euler-Poisson's equations with $\beta = \beta' = -\frac{1}{2}$ in ψ and $\frac{1}{2}$ in ϕ. The Riemann function corresponding to ψ can be shown to be

$$\Psi(\lambda,\mu;\ \lambda_0,\mu_0) = \frac{(\mu_0 - \lambda)^{1/2}(\mu - \lambda_0)^{1/2}}{(\mu - \lambda)}\ {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2};\ 1;\ \sigma\right) \tag{54}$$

where λ_0 $\left(\text{or } \bar{\Omega}_0 + \theta_0\right)$ and μ_0 correspond to $\Omega_0 + \frac{a'}{b'}$ and θ_0, the point of interest, ${}_2F_1$ is the hypergeometrical function, and

$$\sigma = \frac{(\lambda - \lambda_0)(\mu - \mu_0)}{(\lambda - \mu_0)(\mu - \lambda_0)}$$

It becomes too cumbersome to write the Riemann function in terms of the original coordinates Ω and θ. This approximation is shown at $\Omega = \Omega_\infty$ in figure 12. Since the Cauchy data are difficult to assign to the potential the corresponding Riemann function becomes useless. Therefore, with the above Riemann function Ψ the stream function ψ at a point $\left(\Omega_0,\theta\right)$ can be found by the Riemann method, if the Cauchy data are sufficiently given. In general, even if the Cauchy data are sufficiently known in the hodograph, the Riemann method is rather difficult to apply.

Second-order approximation.- Following the same procedure as before, it is not difficult to show that

$$\frac{\partial^2\psi}{\partial\bar{\Omega}^2} - \frac{\partial^2\psi}{\partial\theta^2} - \left(\frac{1}{\bar{\Omega} + 1} + \frac{1}{\bar{\Omega} - 1}\right)\frac{\partial\psi}{\partial\bar{\Omega}} = 0 \tag{55a}$$

$$\frac{\partial^2\phi}{\partial\bar{\Omega}^2} - \frac{\partial^2\phi}{\partial\theta^2} + \left(\frac{1}{\bar{\Omega} + 1} + \frac{1}{\bar{\Omega} - 1}\right)\frac{\partial\phi}{\partial\bar{\Omega}} = 0 \tag{55b}$$

where

$$\bar{\Omega} = \frac{2c'}{\left[(b')^2 - 4a'c'\right]^{1/2}}\left(\Omega - \frac{b'}{2c'}\right) \qquad \left((b')^2 > 4a'c'\right) \quad (56)$$

$$c' = \left[\frac{d^2(-K)^{1/2}}{d\Omega^2}\right]_{\Omega=\Omega_\infty} \quad (57)$$

The Riemann function of ψ is under investigation, and is not so easy to obtain unless a long series solution is adopted. However, the particular solutions are of Stratton's type, but are not given here.

Some better approximation of the coefficient $\frac{d}{d\Omega}\left[\log_e (-K)^{1/2}\right]$ can be made, but the solution to the resulting equation will be more difficult to obtain.

Comments on above approximations.- The above approximations become worse as $\Omega_\infty \longrightarrow 0$ or $M \longrightarrow 1$. Even at very large Mach numbers, they are not very optimistic approximations.

3.3 - Approximations to Transonic Canonical Forms - Equations (30a) and (30b)

Take the transonic canonical forms of equations (30a) and (30b)

$$\frac{\partial^2\psi}{\partial\epsilon^2} - \epsilon\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{2}\frac{\partial\psi}{\partial\epsilon}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right] = 0 \quad (30a)$$

$$\frac{\partial^2\phi}{\partial\epsilon^2} - \epsilon\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{2}\frac{\partial\phi}{\partial\epsilon}\frac{d}{d\epsilon}\left[\log_e(-K\epsilon)\right] = 0 \quad (30b)$$

Following Guderley (reference 30) introduce the variable

$$\zeta = \left(\frac{3}{2}\right)^3 \frac{\theta^2}{\epsilon^3} = \frac{\theta^2}{\Omega^2} \qquad \text{(supersonic)} \tag{58a}$$

$$\zeta = \left(\frac{3}{2}\right)^3 \frac{\theta^2}{\epsilon^3} = \frac{\theta^2}{\omega^2} \qquad \text{(subsonic)} \tag{58b}$$

It is interesting to note that $\zeta > 0$ in the supersonic side and $\zeta < 0$ in the subsonic side of the hodograph plane. The equation $\zeta = 1$ corresponds to the pair of characteristics starting at $\theta = 0$ on the sonic line. The independent variable θ can be eliminated in equation (30a) and equation (30b). Since the two equations are equivalent in behavior, only equation (30a) is treated here. Thus there is obtained

$$\zeta(\zeta - 1) \frac{\partial^2\psi}{\partial\zeta^2} - \frac{2}{3}\epsilon\zeta \frac{\partial^2\psi}{\partial\zeta\,\partial\epsilon} + \frac{\epsilon^2}{9}\frac{\partial^2\psi}{\partial\epsilon^2} + \frac{\partial\psi}{\partial\zeta}\left\{-\frac{1}{2} + \frac{4}{3}\zeta - \frac{\epsilon\zeta}{6}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]\right\} +$$

$$\frac{\partial\psi}{\partial\epsilon}\left\{\frac{\epsilon^2}{18}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]\right\} = 0 \tag{59}$$

Now examine what conditions must be imposed on $\epsilon \frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]$ so that the variables ζ and ϵ are separable in the above equation. First, assume

$$\psi(\epsilon,\zeta) = \psi_1(\epsilon)\psi_2(\zeta) \tag{60}$$

Equation (59) yields

$$\zeta(1 - \zeta)\frac{d^2\psi_2}{d\zeta^2} + \left(\frac{1}{2} - \left\{\frac{1}{3} - \frac{2}{3}\frac{\epsilon}{\psi_1}\frac{d\psi_1}{d\epsilon} - \frac{\epsilon}{6}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right] + 1\right\}\zeta\right)\frac{d\psi_2}{d\zeta} -$$

$$\left\{\frac{1}{9}\frac{\epsilon^2}{\psi_1}\frac{d^2\psi_1}{d\epsilon^2} + \frac{1}{9}\frac{\epsilon}{\psi_1}\frac{d\psi_1}{d\epsilon}\,\frac{\epsilon}{2}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]\right\}\psi_2 = 0 \tag{61}$$

where $K = \left(\frac{\rho_o}{\rho}\right)^2 \left(1 - M^2\right)$ is a known function of ϵ. In general, the above equation cannot be solved unless the following conditions are imposed:

$$\frac{1}{3} - \frac{2\epsilon}{3\psi_1} \frac{d\psi_1}{d\epsilon} - \frac{\epsilon}{6} \frac{d}{d\epsilon}\left[\log_e \left(\frac{-K}{\epsilon}\right)\right] = \alpha_s + \beta_s \tag{62a}$$

and

$$\frac{1}{9} \frac{\epsilon^2}{\psi_1} \frac{d^2\psi_1}{d\epsilon^2} + \frac{1}{9} \frac{\epsilon}{\psi_1} \frac{d\psi_1}{d\epsilon} \frac{\epsilon}{2} \frac{d}{d\epsilon}\left[\log_e \left(\frac{-K}{\epsilon}\right)\right] = \alpha_s\beta_s \tag{62b}$$

where α_s and β_s are individual constants to be chosen later. Equation (61) can then be written as

$$\zeta(1 - \zeta) \frac{d^2\psi_2}{d\zeta^2} + \left[\frac{1}{2} - \left(1 + \alpha_s + \beta_s\right)\zeta\right]\frac{d\psi_2}{d\zeta} - \alpha_s\beta_s\psi_2 = 0 \tag{63}$$

which is a hypergeometric equation. Its singularities are $\zeta = 0$, 1, and ∞. The general solution about $\zeta = 0$ is

$$\psi_2(\zeta) = \left.\begin{array}{l} {}_2F_1\left(\alpha_s, \beta_s; \frac{1}{2}; \zeta\right) \\ \\ {}_2F_1\left(\alpha_s + \frac{1}{2}, \beta_s + \frac{1}{2}; \frac{3}{2}; \zeta\right)\zeta^{1/2} \end{array}\right\} \left(|\zeta| < 1\right) \tag{64a}$$

The solution about $\zeta = 1$ is

$$\psi_2(\zeta) = \left.\begin{array}{l} (1 - \zeta)^{\frac{1}{2} - \alpha_s - \beta_s}\ {}_2F_1\left(\frac{1}{2} - \alpha_s, \frac{1}{2} - \beta_s; \frac{1}{2} - \alpha_s - \beta_s + 1; 1 - \zeta\right) \\ \\ \zeta^{1/2}(1 - \zeta)^{\frac{1}{2} - \alpha_s - \beta_s}\ {}_2F_1\left(1 - \alpha_s, 1 - \beta_s; \frac{1}{2} - \alpha_s - \beta_s + 1; 1 - \zeta\right) \end{array}\right\} \left(|1 - \zeta| < 1\right) \tag{64b}$$

Following Kummer (reference 61), all the 24 solutions about the 3 singular points $\zeta = 0$, 1, and ∞ can be obtained. The triple-valued behavior of $\psi_2(\zeta)$ for $\zeta = 0$ can be shown easily. This agrees with the results of Lighthill, Guderley, and Carrier. Under the imposed condition, it is not difficult to show that

$$\psi_1(\epsilon) = \frac{\text{Constant}}{\epsilon^{3\alpha_s}\left[1 - c_s\epsilon^{3(\beta_s-\alpha_s)}\right]} \tag{65}$$

where $\alpha_s \neq \beta_s$ and c_s is a free constant to be chosen later. Also, if by definition $z_s = \frac{\epsilon}{2}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]$, its approximate value $\tilde{z}_s$ can be calculated from equations (62a) and (62b):

$$\tilde{z}_s = 1 - 3(\beta_s - \alpha_s)\frac{1 + c_s\epsilon^{3(\beta_s-\alpha_s)}}{1 - c_s\epsilon^{3(\beta_s-\alpha_s)}} \tag{66}$$

The exact value of z_s can be determined from equation (25a) with $f(\epsilon) = \epsilon$ and $K = \left(\frac{\rho_o}{\rho}\right)^2\left(1 - M^2\right)$.

It is necessary now to choose the approximate variable $\tilde{z}_s$ so as to have the same ordinate and slope as z_s at $\epsilon = 0$ (or $M = 1$) if the flow at sonic speed is of particular interest. It can be shown that, at $\epsilon = 0$,

$$z_s = 0 \tag{67a}$$

$$\frac{dz_s}{d\epsilon} = \frac{d}{d\epsilon}\left\{\frac{\epsilon}{2}\frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)\right]\right\} \approx 1.05 \tag{67b}$$

These conditions determine

$$\tilde{z}_s = -\frac{2c_s\epsilon}{1 - c_s\epsilon} \tag{68a}$$

$$\beta_s = \alpha_s + \frac{1}{3} \tag{68b}$$

where $c_s \approx -0.525$. With the approximation to equation (30a), the approximate differential equation becomes

$$\frac{\partial^2\psi}{\partial\epsilon^2} - \epsilon\frac{\partial^2\psi}{\partial\theta^2} - \frac{2c_s}{1 - c_s\epsilon}\frac{\partial\psi}{\partial\epsilon} = 0 \tag{69}$$

The singular solutions of this equation are

$$\psi(\epsilon,\theta) = \left.\begin{array}{l} \epsilon^{-3\alpha_s}\left(1 - c_s\epsilon\right)^{-1} {}_2F_1\left(\alpha_s,\alpha_s + \frac{1}{3}; \frac{1}{2}; \frac{9\theta^2}{4\epsilon^3}\right) \\ \\ \epsilon^{-3\alpha_s}\left(1 - c_s\epsilon\right)^{-1} \dfrac{3\theta}{2\epsilon^{3/2}} {}_2F_1\left(\alpha_s + \frac{1}{2},\alpha_s + \frac{5}{6}; \frac{3}{2}; \frac{9\theta^2}{4\epsilon^3}\right) \end{array}\right\} \tag{70}$$

where α_s should be chosen to avoid the limiting line in the physical flow pattern. It should be noted that equation (69) and its equivalent equation of ϕ (not shown here) are one order higher in approximation than the Tricomi equation. In other words, the Tricomi equation is equivalent to taking $z_s = 0$ for all values of ϵ. For negative values of α_s, the singularity of ψ is at $\epsilon = \frac{1}{c_s}$. If α_s is positive, there are two singularities of ψ, one at $\epsilon = 0$ and other at $\epsilon = \frac{1}{c_s}$.

4 - DIFFERENT APPROXIMATIONS TO CHAPLYGIN'S SECOND DIFFERENTIAL EQUATION AND THEIR SOLUTIONS

From equations (10a) and (10b),

$$\frac{\partial^2 \psi}{\partial \sigma^2} + K \frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{71a}$$

$$\frac{\partial^2 \phi}{\partial \sigma^2} + K \frac{\partial^2 \phi}{\partial \theta^2} - \frac{d}{d\sigma}\left(\log_e K\right)\frac{\partial \phi}{\partial \sigma} = 0 \tag{71b}$$

where $K = \left(\frac{\rho_o}{\rho}\right)^2\left(1 - M^2\right)$ is an implicit function of σ. The exact solutions are difficult to obtain. Chaplygin (reference 1) in his researches on subsonic gas jets chose $K = 1$ so that his equations become Laplace equations. As shown in figure 3 this is a reasonable approximation if the maximum Mach number in the jet is much less than unity, because $K = 1$ is actually the asymptote for the true $K - \sigma$ curve at $M = 0$. Kármán (reference 7) and Tsien (reference 52) extend the idea to the linear approximation of the pressure-volume relation corresponding to the free-stream condition, also obtaining a Laplace equation and achieving many fruitful results of technical importance. In the transonic flow, of course, such approximations cannot be applied. Von Kármán (reference 29) in 1947 gave the approximate differential equations which are valid when the flow velocity is in the neighborhood of sonic velocity and when the body in the flow is thin. He also found the transonic similarity law which gives a satisfactory prediction of the wave drag for thin bodies. In the hodograph plane, the approximate differential equations are of the Tricomi type. The concept of the similarity law has been further discussed by Kaplan (reference 39) and Guderley (reference 31) and extended by Tsien to hypersonic flow. The main contribution of the transonic similarity law to the hodograph method is: Instead of investigating the flow about a given thin body, the flow of a body with the same thickness distribution as the given one but of vanishing thickness ratio can be investigated. Because of the vanishing thickness ratio, the boundary conditions can be simplified in the hodograph plane and are shown to be consistent with the approximation applied to the differential equations. Under these conditions, the problem becomes a boundary-value problem of the type studied by

Tricomi. It is unnecessary to use associated boundary conditions from the incompressible flow as done by Chaplygin. After the transonic flow of the body with vanishing thickness ratio is obtained the transonic similarity law may be applied to find the aerodynamic behavior of the given thin body.

The success of the above approximation encourages the author to seek some higher-order approximations. First take a look at the $K - \sigma$ curve shown in figure 3. It has two asymptotes: One is $K = 1$ and the other is $\sigma = -0.2513$. It is a monotonic increasing function of σ within the range $-0.2513 < \sigma < \infty$; at $\sigma = 0$, $K = 0$. It is understood that $\sigma > 0$ corresponds to $M < 1$ and $\sigma < 0$ to $M > 1$. The main interest of the investigation lies in transonic flow. Therefore, if possible, the approximation should be so chosen as to maintain the behavior of the exact $K - \sigma$ curve at and near the sonic velocity ($\sigma = 0$) and at the same time preserve the asymptotic behavior of the exact curve as $M \longrightarrow 0$ $(\sigma \longrightarrow \infty)$ and $M \longrightarrow \infty$ $(\sigma \longrightarrow -0.2513)$.

Just as important, the approximation equation should possess solutions within the reach of classical mathematical analysis. With these few criterions in mind, it can be seen that the Taylor series expansion of K about $\sigma = 0$ is not a favorable choice, although the first approximation to be shown below is of this nature.

4.1 - First-Order Approximation in Neighborhood of Sonic Speed

Frankl (reference 11) in 1945 obtained the Tricomi equation from equations (71) by letting $K \approx \widetilde{K}_1 = a\sigma$ and consequently

$$\frac{\partial^2\psi}{\partial\sigma^2} + a\sigma \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{72a}$$

$$\frac{\partial^2\phi}{\partial\sigma^2} + a\sigma \frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{\sigma}\frac{\partial\phi}{\partial\sigma} = 0 \tag{72b}$$

where $a = \left(\frac{dK}{d\sigma}\right)_{\sigma=0} \approx 9.42$ and $\sigma = \int_q^{a^*} \frac{\rho}{\rho_0}\frac{1}{q}\,dq$ as given in the earlier definition. The comparison of the approximation $\widetilde{K}_1$ with the exact value of K is shown in figure 3. From the figure the range of validity of this approximation is very narrow, only in the immediate neighborhood of the sonic velocity a^*. The solutions of these equations have been fully investigated by Tricomi in his famous thesis (reference 53). Carrier and Ehlers (reference 57) and also Weinstein (reference 40) have further investigated the singular solutions of these equations; Tomotika and Tamada (reference 33) also made interesting contributions to the singular and fundamental solutions. Recently Guderley (reference 31) made an extensive study of the singular solutions and showed a very important singular solution corresponding to a family of airfoils at sonic speed. Guderley and Yoshihara (reference 36) gave the flow over a wedge airfoil at Mach number 1. They employ an elegant method of attack in solving the problem.

In the early days, Euler and Darboux obtained the fundamental solutions of the same equation in the pure elliptic and in the pure hyperbolic domains as pointed out by Tricomi (reference 53). The importance of Tricomi's work is the recognition of the differential equation in the mixed domain, that is, partly elliptic and partly hyperbolic, and the proof of the existence and uniqueness of the solutions in such a mixed domain. Following the footsteps of Tricomi, Frankl (reference 11) has shown the existence and uniqueness of the solution of Chaplygin's second differential equation in the mixed domain, particularly on two problems, one being the detached shock wave of a wedge and the other being the supersonic jet from an inclined-walled vessel.

4.2 - Second-Order Approximation

From the comparison of the first-order approximation and the exact curve in figure 3 it is immediately apparent that some improved approximation should have a wider range of validity than that given by the linear approximation. The usual technique of taking higher-order terms in the Taylor series would not be particularly helpful because they would not improve the asymptotic behavior for large negative values of K and for large values of σ. Therefore, choose

$$K(\sigma) = \tilde{K}_2(\sigma) = \frac{a\sigma}{1 + c\sigma} \tag{73}$$

where $a = \left(\frac{d\tilde{K}_2}{d\sigma}\right) = \left(\frac{dK}{d\sigma}\right) = 9.42$ at $\sigma = 0$ and c can be chosen in any one of three ways:

(1) $\left(\frac{d^2\tilde{K}_2}{d\sigma^2}\right) = \left(\frac{d^2K}{d\sigma^2}\right)$ at $\sigma = 0$, favorable for transonic range

(2) $c = a$ so that $\tilde{K}_2 = K \longrightarrow 1$ exactly as $\sigma \longrightarrow \infty$, favorable for transonic and subsonic range

(3) $c = -1/0.2513$ so that $\tilde{K}_2 = K \longrightarrow -\infty$ exactly as $\sigma \longrightarrow -0.2513$, favorable for transonic and supersonic range

It is interesting to note that, in equation (73), $\tilde{K}_2 \longrightarrow \frac{a}{c}$, a finite value, as $\sigma \longrightarrow \infty$, and $\tilde{K}_2 \longrightarrow \infty$ as $\sigma \longrightarrow -\frac{1}{c}$. Thus, for any one of the three choices, $\tilde{K}_2$ always has two desirable asymptotes. Therefore, the approximation should be fairly good in the subsonic range and supersonic range. Of course, if Chaplygin's procedure of using boundary conditions similar to those for an incompressible flow is followed, the second choice is a favorable one. Cases (1) and (2) are shown in the figure. Both seem good in the transonic range. Case (3) is not shown. The only known second-order approximation is given by Loewner (reference 73). It is also shown in figure 3 for comparison.

With the above approximation equations (71a) and (71b) become

$$\frac{\partial^2\psi}{\partial\sigma^2} + \frac{a\sigma}{1 + c\sigma}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{74a}$$

$$\frac{\partial^2\phi}{\partial\sigma^2} + \frac{a\sigma}{1 + c\sigma}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\sigma(1 + c\sigma)}\frac{\partial\phi}{\partial\sigma} = 0 \tag{74b}$$

Equation (74a) can be solved as follows:

Assume the variables in $\psi(\sigma,\theta)$ separable and let

$$\psi(\sigma,\theta) = \psi_1{}^{(n)}(\sigma)\psi_2{}^{(n)}(\theta)$$

where the positive integer n is introduced to characterize the nature of the solution

$$\psi_2^{(n)}(\theta) = \begin{cases} c_3\theta + c_4 & (n = 0) \\ \begin{matrix}\cos\\ \sin\end{matrix}\,(n\lambda\theta) & (n \neq 0) \end{cases}$$

where the positive constant λ is so chosen that $\begin{matrix}\sin\\ \cos\end{matrix}\,(n\lambda\theta)$ is periodic for any fixed interval of θ. For $n \neq 0$, $\psi_1^{(n)}(\sigma)$ satisfies the equation

$$\frac{d^2\psi_1^{(n)}}{d\sigma^2} - \frac{n^2\lambda^2 a\sigma}{1 + c\sigma}\psi_1^{(n)} = 0 \tag{75}$$

Introduce a new independent variable $z = 2n\lambda\left(\frac{a}{c}\right)^{1/2}\left(\sigma + \frac{1}{c}\right)$. Obviously, in the range $-\frac{1}{c} \leqq \sigma \leqq \infty$, the corresponding range for z is $0 \leqq z \leqq \infty$. Equation (75) yields

$$\frac{d^2\psi_1^{(n)}(z)}{d\sigma^2} + \left(-\frac{1}{4} + \frac{k}{z}\right)\psi_1^{(n)}(z) = 0 \tag{76}$$

which is a particular case of the Whittaker equation with $m^2 = \frac{1}{4}$ (for a reason to be shown later). Here $k = \frac{n\lambda}{2c}\left(\frac{a}{c}\right)^{1/2} > 0$.

Before the choice of the solution is made from the known results of classical mathematics, a clear understanding of the nature of the desired solution when $\sigma \longrightarrow \infty$ or when the flow becomes incompressible is necessary. As $\sigma \longrightarrow \infty$, $z \longrightarrow \infty$ and $\psi_1^{(n)}(z)$ must behave like $e^{\pm\frac{z}{2}}$ from the limiting case of equation (76). Furthermore $\psi_1^{(n)}(z)\begin{Bmatrix}\sin n\lambda\theta\\ \cos n\lambda\theta\end{Bmatrix}$ will be just a single term of the convergent infinite series which represents $\psi(\sigma,\theta)$, to be shown later. The term $\psi_1^{(n)}(z)$ should remain finite when $\sigma \longrightarrow \infty$. Therefore, $\psi_1^{(n)}(z) \sim e^{-\frac{z}{2}}$ as $\sigma \longrightarrow \infty$ is necessary.

With the desirable asymptotic behavior of $\psi_1^{(n)}(z)$ at large values of σ in mind, the only choice is

$$\psi_1^{(n)}(z) = W^{(n)}_{k,-\frac{1}{2}}(z)$$

whose integral representation (see reference 61) is

$$W^{(n)}_{k,-\frac{1}{2}}(z) = \frac{\Gamma(k+1)e^{-\frac{z}{2}}z^k}{2\pi i}\int_\infty^{(0+)}(-t)^{-k-1}\left(1+\frac{t}{z}\right)^{k-1}e^{-t}\,dt \tag{77}$$

where the path of integration is a contour in the complex t-plane starting from ∞ just above the real axis, encircling the origin in the positive direction (counterclockwise), and returning to the starting point just below the real axis. In general, for $k > 0$ there are two branch points, one at $t = 0$ and the other at $t = -z$, a point on the negative real axis (since $z > 0$). Thus the path of integration must be chosen so that it will not encircle the branch point $t = -z$.

The asymptotic expansion for large values of σ (not simultaneously of large values of n) is

$$W^{(n)}_{k,-\frac{1}{2}}(z) \sim e^{-\frac{z}{2}}z^k\left\{1+\sum_{s=1}^{\infty}\frac{\left[\frac{1}{4}-\left(k-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}-\left(k-\frac{3}{2}\right)^2\right]\cdots\left[\frac{1}{4}-\left(k-s+\frac{1}{2}\right)^2\right]}{s!\,z^s}\right\} \tag{78}$$

For the case of large values of n, the asymptotic expansion of $W^{(n)}_{k,-\frac{1}{2}}(z)$ is much more complicated owing to the fact that both k and z, being proportional to σ, become large simultaneously. Very little information on such expansions is available.

For numerical calculation, the corresponding series solution is desirable. Jeffreys (reference 71) condenses early developments of Whittaker, Goldstein, and Stoneley and puts into compact form the relation between the Whittaker function and the confluent hypergeometric function. The following discussion closely parallels his work except for some changes in notation. For $2m = \pm\mu$ (μ is a positive integer), the Whittaker function can be represented by a combination of Kummer's series when the limiting value is taken. Thus,

$$W_{k,\pm\mu}(z) = \lim_{2m\to\pm\mu}\left[\frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2} - m - k\right)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2} + m - k\right)} M_{k,-m}(z)\right]$$

where

$$M_{k,\pm m}(z) = e^{-\frac{z}{2}} z^{\pm m+\frac{1}{2}} \frac{\Gamma(\pm 2m + 1)}{\Gamma\left(\pm m - k + \frac{1}{2}\right)} \sum_{s=0}^{\infty} \frac{\Gamma\left(\pm m - k + \frac{1}{2} + s\right)}{\Gamma(\pm 2m + 1 + s)} \frac{z^s}{s!}$$

$$= e^{-\frac{z}{2}} z^{\pm m+\frac{1}{2}} {}_1F_1\left(\pm m - k + \frac{1}{2}, \pm 2m + 1;\ z\right)$$

${}_1F_1(\alpha,\gamma;\ z)$ being the confluent hypergeometric function. $\left(\alpha = m - k + \frac{1}{2},\ \ \gamma = -2m + 1\right.$ according to Jeffreys' notation.$\left.\right)$

$$W_{k,\pm\mu}(z) = \lim_{2m\to\pm\mu}\left[e^{-\frac{z}{2}} z^{-m+\frac{1}{2}} U\left(-m - k + \frac{1}{2}, -2m + 1;\ z\right)\right]$$

where

$$U\left(-m - k + \frac{1}{2}, -2m + 1;\ z\right) = \frac{\Gamma(-2m)}{\Gamma\left(-m - k + \frac{1}{2}\right)}\ {}_1F_1\left(m - k + \frac{1}{2}, 2m + 1;\ z\right) +$$

$$\frac{\Gamma(2m)z^{2m}}{\Gamma\left(m - k + \frac{1}{2}\right)}\ {}_1F_1\left(-m - k + \frac{1}{2}, -2m + 1;\ z\right)$$

He gives the solution of the limiting case $2m \longrightarrow \pm\mu$. For the present case if m is chosen equal to $-\frac{1}{2}$ ($\gamma = 2$ in his notation), there can be written

$$W_{k,-\frac{1}{2}}(z) = e^{-\frac{z}{2}}\, z\left[U_1(z) + U_2(z)\right] \tag{79a}$$

where

$$U_1(z) = \frac{1}{\Gamma(1 - k)}\, z^{-1} \tag{79b}$$

$$U_2(z) = \frac{1}{\Gamma(-k)}\ {}_1F_1(1 - k, 2,;\ z)\left[\log_e z - F(1) - F(0) + F(-k)\right] + U_3 \tag{79c}$$

The function $F(\zeta)$ is called the digamma function and is generally represented by $\psi(\zeta + 1)$ (reference 74).

$$F(\zeta) = \frac{d}{d\zeta}\left[\log_e \Gamma(\zeta + 1)\right]$$

$$= \lim_{n \longrightarrow \infty}\left(\log_e n - \frac{1}{\zeta + 1} - \frac{1}{\zeta + 2} - \frac{1}{\zeta + 3} - \cdot \cdot \cdot \frac{1}{\zeta + n}\right) \tag{80}$$

($F(0)$ = Euler constant $(-\gamma) = -0.57722.$)

For the present case,

$$U_3(z) = \sum_{s=1}^{\infty} \frac{z^s}{\Gamma(-k)} \frac{(1 - k)(2 - k) \ . \ . \ . \ (s - k)}{s!(s + 1)!} \left(\sum_{r=1}^{s} \frac{1}{r - k} - 2\sum_{r=1}^{s} \frac{1}{r} + 1 - \frac{1}{s + 1}\right)$$

$$= \sum_{s=1}^{\infty} \frac{\Gamma(1 + s - k)}{\Gamma(-k)\Gamma(1 - k)} z^s \left(\sum_{r=1}^{s} \frac{1}{r - k} - 2\sum_{r=1}^{s} \frac{1}{r} + 1 - \frac{1}{s + 1}\right)$$

$$(k \neq \text{integer}) \qquad (81)$$

Then, the series solution of equation (76) is

$$\psi(\sigma,\theta) = \left(c_1\sigma + c_2\right)\left(c_3\theta + c_4\right) + \sum_{n=1}^{\infty} A_n W^{(n)}_{k,-\frac{1}{2}} (z) \sin\left(n\lambda\theta + \alpha_n\right) \qquad (82)$$

where c_1, c_2, c_3, c_4, A_n, and α_n are constants to be determined from the given boundary conditions or from the boundary conditions associated with an incompressible flow as in Chaplygin's method, if necessary. Of course, the convergence of the series must be established for the particular problem in order to be sure the above representation is correct.

4.3 - Third-Order Approximation

The $K - \sigma$ relation can be approximated more closely by assuming an analytic function of σ in the range $-\frac{1}{c} < \sigma < \infty$ of the form

$$K = \tilde{K}_3 = \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} \qquad (83)$$

such that

(1) $\tilde{K}_3 = K = 0$ at $\sigma = 0$

(2) $\dfrac{d\tilde{K}_3}{d\sigma} = \dfrac{dK}{d\sigma} = a$ at $\sigma = 0$

(3) $\dfrac{d^2\tilde{K}_3}{d\sigma^2} = \dfrac{d^2K}{d\sigma^2}$ at $\sigma = 0$

(4) $\tilde{K}_3(\sigma)$ also possesses two asymptotes, one of which can be made to coincide with an exact asymptote of $K(\sigma)$.

The conditions (3) and (4) determine the constants b and c. The above approximation with $\tilde{K}_3 \longrightarrow \dfrac{ab}{c^2} = 1$ as $\sigma \longrightarrow \infty$ is shown in figure 13. It checks very well with the exact value of K for the range from subsonic through transonic up to supersonic regimes. The other asymptote, $\sigma = -0.2583$ as $\tilde{K}_3 = -\infty$, does not differ greatly from $\sigma = -0.2513$ for the exact value of K. There is another important advantage of this choice, because the boundary conditions of incompressible flows can be borrowed as in the Chaplygin procedure.

Introducing equation (83) into equations (71a) and (71b),

$$\frac{\partial^2\psi}{\partial\sigma^2} + \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{84a}$$

and

$$\frac{\partial^2\phi}{\partial\sigma^2} + \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{(2b - c)\sigma + 1}{\sigma(1 + b\sigma)(1 + c\sigma)}\frac{\partial\phi}{\partial\sigma} = 0 \tag{84b}$$

To solve equation (84a), assume that the variables are separable and

$$\psi^{(n)}(\sigma,\theta) = \psi_1^{(n)}(\sigma)\psi_2^{(n)}(\theta)$$

where the superscript (n) is used to show each solution is related to n, a positive integer,

$$\psi_2^{(n)}(\theta) = \begin{cases} \begin{matrix}\sin\\ \cos\end{matrix}\,(n\lambda\theta) & (n \neq 0) \\ c_3\theta + c_4 & (n = 0) \end{cases} \tag{85}$$

and $\psi_1^{(n)}(\sigma)$ satisfies

$$\frac{d^2\psi_1^{(n)}}{d\sigma^2} - \frac{n^2\lambda^2 a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}\psi_1^{(n)} = 0 \tag{86}$$

With the introduction of a new independent variable $z = 2n\lambda \frac{\sqrt{ab}}{c^2}(1 + c\sigma)$, equation (86) can be transformed to the well-known Whittaker equation

$$\frac{d^2\psi_1^{(n)}}{d\sigma^2} - \left(\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right)\psi_1^{(n)} = 0 \tag{87}$$

where

$$k = -n\,\frac{c - 2b}{2c^2}\sqrt{\frac{a}{b}} < 0 \qquad (c > 2b) \tag{88a}$$

$$m = \sqrt{\frac{1}{4} - \frac{n^2\lambda^2 a(c - b)}{c^4}} \qquad \begin{matrix} \left(\text{imaginary if } \dfrac{n^2\lambda^2 a(c - b)}{c^4} > \dfrac{1}{4}\right) \\ \left(\text{real if } \dfrac{n^2\lambda^2 a(c - b)}{c^4} < \dfrac{1}{4}\right) \end{matrix} \tag{88b}$$

The fundamental system of solutions of equation (86) is

$$\psi_1^{(n)}(z) = \begin{cases} C_n W_{k,m}^{(n)}(z) + D_n W_{-k,m}^{(n)}(-z) & (n \neq 0) \\ c_1\sigma + c_2 & (n = 0) \end{cases} \tag{89}$$

where C_n and D_n are constants to be determined.

The Whittaker functions can be expressed in terms of the confluent hypergeometric functions and Kummer's functions if $m \neq \pm\frac{1}{2}$ as follows:

$$W_{k,m}^{(n)}(z) = \frac{\Gamma(-2m)}{\Gamma\left(-m - k + \frac{1}{2}\right)} M_{k,m}^{(n)}(z) + \frac{\Gamma(2m)}{\Gamma\left(m + k + \frac{1}{2}\right)} M_{k,-m}^{(n)}(z) \tag{90a}$$

and

$$W_{-k,m}^{(n)}(-z) = \frac{\Gamma(-2m)}{\Gamma\left(-m + k + \frac{1}{2}\right)} M_{-k,m}^{(n)}(-z) + \frac{\Gamma(2m)}{\Gamma\left(m - k + \frac{1}{2}\right)} M_{-k,-m}^{(n)}(-z)$$

$$\left(\left|\arg(-z)\right| < \frac{3}{2}\pi\right) \tag{90b}$$

where the Kummer's series are

$$M_{k,\pm m}^{(n)}(z) = e^{-\frac{z}{2}} z^{\pm m + \frac{1}{2}} \frac{\Gamma(\pm 2m + 1)}{\Gamma\left(\pm m - k + \frac{1}{2}\right)} \sum_{s=0}^{\infty} \frac{\Gamma\left(\pm m - k + \frac{1}{2} + s\right) z^s}{\Gamma(\pm 2m + 1 + s)s!}$$

$$= e^{-\frac{z}{2}} z^{\pm m + \frac{1}{2}} {}_1F_1\left(\pm m - k + \frac{1}{2}, \pm 2m + 1;\ z\right)$$

$$M_{-k,\pm m}^{(n)}(-z) = e^{\frac{z}{2}} z^{\pm m + \frac{1}{2}} \frac{\Gamma(\pm 2m + 1)}{\Gamma\left(\pm m + k + \frac{1}{2}\right)} \sum_{s=1}^{\infty} \frac{\Gamma\left(\pm m + k + \frac{1}{2} + s\right)}{\Gamma(\pm 2m + 1 + s)} \frac{(-z)^s}{s!}$$

$$= e^{\frac{z}{2}} z^{\pm m + \frac{1}{2}} {}_1F_1\left(\pm m + k + \frac{1}{2}, \pm 2m + 1;\ -z\right)$$

Besides, the integral solutions are much more general, and the Whittaker functions can be defined uniquely whether or not $m = \pm\frac{1}{2}$.

$$W^{(n)}_{k,m}(z) = -\frac{1}{2\pi i}\,\Gamma\left(-m + k + \frac{1}{2}\right)e^{-\frac{z}{2}}\,z^{k}\int_{\infty}^{(0+)}(-t)^{m-k-\frac{1}{2}}\left(1 + \frac{t}{z}\right)^{m+k-\frac{1}{2}}e^{-t}\,dt \tag{91a}$$

$$W^{(n)}_{-k,m}(-z) = -\frac{1}{2\pi i}\,\Gamma\left(-m - k + \frac{1}{2}\right)e^{\frac{z}{2}}\,(-z)^{-k}\int_{\infty}^{(0+)}(-t)^{m+k-\frac{1}{2}}\left(1 - \frac{t}{z}\right)^{m-k-\frac{1}{2}}e^{-t}\,dt \tag{91b}$$

In the present case z is real and positive and the contours are so chosen that the second branch point $t = -z$ is excluded. For more details consult reference 61.

Now the question arises whether both or only one of the two Whittaker functions exists in the present solution. This can be determined from considering the incompressible flow as the limiting case as $\sigma \longrightarrow \infty$ $(M \longrightarrow 0)$. Furthermore, $\psi_1^{(n)}(z)\begin{Bmatrix}\sin n\lambda\theta \\ \cos n\lambda\theta\end{Bmatrix}$ is just the nth term of the convergent series solution. In other words $\psi_1^{(n)}(z)$ must be finite as $\sigma \longrightarrow \infty$. Now, from equations (91a) and (91b), the asymptotic expansions for large values of σ are

$$W^{(n)}_{k,m}(z) \sim e^{-\frac{z}{2}}\,z^{k}\left\{1 + \sum_{s=1}^{\infty}\frac{\left[m^2 - \left(k - \frac{1}{2}\right)^2\right]\left[m^2 - \left(k - \frac{3}{2}\right)^2\right]\ldots\left[m^2 - \left(k - s + \frac{1}{2}\right)^2\right]}{s!\,z^{s}}\right\} \tag{92a}$$

and

$$W^{(n)}_{-k,m}(-z) \sim e^{\frac{z}{2}}\,(-z)^{-k}\left\{1 + \sum_{s=1}^{\infty}\frac{\left[m^2 - \left(k + \frac{1}{2}\right)^2\right]\left[m^2 - \left(k + \frac{3}{2}\right)^2\right]\ldots\left[m^2 - \left(k + s - \frac{1}{2}\right)^2\right]}{s!\,(-z)^{s}}\right\} \tag{92b}$$

Equation (92a) gives an exponentially damping function of z and equation (92b) is an exponentially increasing function of z which is divergent as $\sigma \longrightarrow \infty$. Therefore, the only choice is $D_n = 0$. Thus, there can be written

$$\psi(\sigma,\theta) = \left(c_1\sigma + c_2\right)\left(c_3\theta + c_4\right) + \sum_{n=1}^{\infty} \left[\overline{A}_n W_{k,m}^{(n)}(z) \cos n\lambda\theta + \overline{B}_n W_{k,m}^{(n)}(z) \sin n\lambda\theta\right] \tag{93}$$

where c_1, c_2, c_3, c_4, $\overline{A}_n$, $\overline{B}_n$, and λ are determined from the given boundary conditions or from the boundary conditions of incompressible flow, if necessary. If $\overline{A}_n \cos \alpha_n$ and $\overline{B}_n \sin \alpha_n$ are introduced, there can be written

$$\psi(\sigma,\theta) = \left(c_1\sigma + c_2\right)\left(c_3\theta + c_4\right) + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin\left(n\lambda\theta + \alpha_n\right) \tag{94}$$

It is interesting to note that, when $m = \pm\frac{1}{2}$, equation (87) reduces to equation (76). Thus the second-order approximation is just one particular case of the third-order approximation with $b = c$. Of course, when the choice is made that $b = c = 0$, both cases reduce to the first-order approximation, equation (72a).

Along this line of thought, if there is imposed $k = 0$ or $2b = c$, equation (82) becomes

$$\tilde{K} = \frac{a\sigma\left(1 + \frac{c}{2}\sigma\right)}{(1 + c\sigma)^2} \tag{95}$$

The differential equation becomes the normal form of the Bessel equation

$$\frac{d^2\psi_1{}^{(n)}}{dz^2} + \left(-\frac{1}{4} + \frac{\frac{1}{4} - m^2}{z^2}\right)\psi_1{}^{(n)} = 0 \tag{96}$$

where

$$z = 2^{1/2} n\lambda\sqrt{\frac{a}{b}}\left(\frac{1}{c} + \sigma\right) \tag{97a}$$

$$m = \sqrt{\frac{1}{4} - \frac{n^2\lambda^2 a}{2c^3}} \tag{97b}$$

Its solution can be written as

$$\psi_1{}^{(n)}(z) = W_{o,m}^{(n)}(z) = \sqrt{\frac{z}{\pi}} \sec m\pi K_m{}^{(n)}\left(\frac{z}{2}\right)$$

where $K_m{}^{(n)}\left(\frac{z}{2}\right)$ is the modified Bessel function of the second kind with complex order m. The function $K_m{}^{(n)}\left(\frac{z}{2}\right)$ and no others satisfy the requirement of boundedness in value as $\sigma \longrightarrow \infty$. This can be shown by the asymptotic expansion

$$K_m{}^{(n)}\left(\frac{z}{2}\right) \sim \sqrt{\frac{4}{\pi z}}\, e^{-\frac{z}{2}}\left\{1 + \sum_{s=1}^{\infty} \frac{\left[m^2 - \left(\frac{1}{2}\right)^2\right]\left[m^2 - \left(\frac{3}{2}\right)^2\right] \cdot \cdot \cdot \left[m^2 - \left(s - \frac{1}{2}\right)^2\right]}{s!\, z^s}\right\} \tag{98}$$

There is another interesting feature of this approximation if the following conditions are chosen: $a = \frac{dK}{d\sigma}$ at $\sigma \longrightarrow 0$ and c such that $c = \frac{a}{2}$ or $\tilde{K} \longrightarrow 1$ as $\sigma \longrightarrow \infty$. Then,

$$\tilde{K} = \frac{a\sigma\left(1 + \frac{a}{4}\sigma\right)}{\left(1 + \frac{a}{2}\sigma\right)^2} \tag{99}$$

and

$$z = n\lambda\left(\frac{2}{a} + \sigma\right) \tag{100a}$$

$$m = \sqrt{\frac{1}{4} - \frac{4n^2\lambda^2}{a^2}} \tag{100b}$$

The advantage of this is that the boundary conditions can still be brought from the incompressible flow.

The accuracy and usefulness of this third-order approximation is discussed in the last section of the paper.

5 - HYPOTHETICAL GAS LAW CORRESPONDING TO APPROXIMATIONS OF CHAPLYGIN'S SECOND EQUATION

In equation (71a), if $\psi = \psi_1(\sigma)\psi_2(\theta)$ is introduced, the principle of separation of variables yields:

$$\frac{d^2\psi_1}{d\sigma^2} - n^2\lambda^2K(\sigma)\psi_1 = 0 \tag{101}$$

where

$$K = \frac{{\rho_o}^2}{\rho^2}\left(1 - M^2\right) = \frac{1 - \mu_1\tau}{(1 - \tau)^{\mu_1}} \tag{102}$$

$$\sigma = -0.2513 - (1 - \tau)^{1/2}\left[1 - \frac{1 - \tau}{3} + \frac{(1 - \tau)^2}{5}\right] + \tanh^{-1}(1 - \tau)^{1/2} \tag{103}$$

$$\frac{d\sigma}{dq} = -\frac{\rho}{\rho_o}\frac{1}{q} \tag{104}$$

Since $K(\sigma)$ is an implicit function of σ, some approximations of different orders have been made in the preceding section to obtain useful solutions of ψ_1. It is natural to ask what kind of hypothetical gas laws will correspond to the different approximations.

It is understood that σ is just another way of expressing the velocity magnitude; consequently it is defined once and for all time by the exact gas behavior as shown in equation (103). The variable σ is independent of the hypothetical gas law for each approximation shown in the preceding section. The problem now is to find the functional relations of p and ρ/ρ_o in terms of σ as the independent variable. First of all, by definition,

$$\frac{dp}{d\rho} = (a*)^2 \tag{105}$$

and the differential form of Bernoulli's equation gives

$$\frac{dp}{dq} = -\rho q$$

If equation (104) is rewritten

$$\frac{\rho}{\rho_o} = \frac{d}{d\sigma}\left(\log_e q^{-1}\right) \tag{106}$$

it can easily be shown that

$$M^2 = \frac{d}{d\sigma}\left(\frac{\rho}{\rho_o}\right) \tag{107}$$

Consequently, if M^2 is expressed in terms of q^{-1} and its derivatives with respect to σ, equations (106) and (107) yield

$$M^2 = 1 - \frac{q^{-1}\,\dfrac{d^2\left(q^{-1}\right)}{d\sigma^2}}{\left[\dfrac{d\left(q^{-1}\right)}{d\sigma}\right]^2} \tag{108a}$$

On the other hand, equations (102) and (106) give an alternate expression for M^2 in terms of q^{-1} and its derivatives

$$M^2 = 1 - \left(\frac{\rho}{\rho_o}\right)^2 K(\sigma) = 1 - \left[\frac{d}{d\sigma}\left(\log_e q^{-1}\right)\right] K(\sigma) \tag{108b}$$

Equating the above two equations, a differential equation is obtained of q^{-1} with respect to σ,

$$\frac{d^2\left(q^{-1}\right)}{d\sigma^2} - K(\sigma) q^{-1} = 0 \tag{109}$$

which is the same equation as equation (101), if q^{-1} is placed for ψ and $n\lambda$ is set equal to 1. If the solutions of equation (101) are known, the solutions of q can similarly be obtained explicitly in terms of σ.

If now the solution of equation (109) is substituted into equation (106), the density ratio is obtained as a function of σ. Differentiating this expression with respect to σ as given in equation (107), the Mach number is obtained in terms of σ. The derivatives above involve only differentiation but $p(\sigma)$ has to be obtained by the integration

$$p = \int \rho_o q^2 \, d\sigma \tag{110}$$

Of course, if equation (109) can be solved exactly, the relations of ρ/ρ_o and p with respect to σ will coincide with those already obtained from the exact gas law.

Now, introduce different orders of approximation to $K(\sigma)$ in order to solve equation (109). Then, obtain the corresponding approximate solution of q^{-1} in terms of σ called $\tilde{q}^{-1}(\sigma)$. With the term $\tilde{q}^{-1}(\sigma)$, $\rho(\sigma)/\rho_o$ can be obtained from equation (104) and $p(\sigma)$ from equation (110). Then, the approximations can be compared with the exact values, both in terms of σ.

In the preceding section, the best approximation to $K(\sigma)$ is the third approximation

$$\tilde{K}_3 = \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} \tag{111}$$

where a, b, and c are constants already chosen according to certain considerations. Substituting into equation (109), there results the differential equation of $\tilde{q}^{-1}$ corresponding to the approximation of q^{-1},

$$\frac{d^2\left(\tilde{q}^{-1}\right)}{d\sigma^2} - \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}\left(\tilde{q}^{-1}\right) = 0 \tag{112}$$

The proper choice of the solution is

$$\tilde{q}^{-1} = EM_{k,m}(z) + FM_{k,-m}(z) \tag{113}$$

where

$$\left.\begin{aligned} z &= \frac{2\sqrt{ab}}{c^2}(1 + c\sigma) > 0 \\ k &= -\frac{c - 2b}{2c^2}\sqrt{\frac{a}{b}} < 0 \\ m &= \sqrt{\frac{1}{4} - \frac{a(c - b)}{c^4}} > 0 \end{aligned}\right\} \tag{114}$$

The terms $M_{k,m}(z)$ and $M_{k,-m}(z)$ are Kummer's confluent hypergeometric series (reference 61) as shown in equations (90a) and (90b) and E and F are arbitrary constants to be determined from the following boundary conditions:

(a) At $\sigma = 0$, $\tilde{q} = q = a*$

(b) At $\sigma = 0$, $\dfrac{d(\tilde{q}^{-1})}{d\sigma} = \dfrac{dq^{-1}}{d\sigma} = \left(\dfrac{\rho_o}{\rho q}\right)_{\sigma=0} = \dfrac{1}{a*}\left(\dfrac{\gamma + 1}{2}\right)^{\beta}$

In addition, q^{-1} must go to ∞ as $\sigma \longrightarrow \infty$ $(M \sim q = 0)$.

The above are the correct boundary conditions because it is intended to choose the hypothetical gas law to agree with the exact one at sonic velocity $(\sigma = 0)$ to the highest possible order of approximation and to preserve the asymptotic behavior as $\sigma \longrightarrow \infty$.

If there is introduced $z = z_o = \dfrac{2\sqrt{ab}}{c^2}$ at $\sigma = 0$,

$$E = \frac{1}{a*z_o W(z_o)}\left[z_o M'_{k,-m}(z_o) - \left(\frac{\gamma + 1}{2}\right)^{\beta} \frac{1}{c} M_{k,-m}(z_o)\right] \tag{115}$$

$$F = \frac{1}{a*z_o W(z_o)}\left[\left(\frac{\gamma + 1}{2}\right)^{\beta} \frac{1}{c} M_{k,m}(z_o) - z_o M'_{k,m}(z_o)\right] \tag{116}$$

where $W(z_o) = M_{k,m}(z_o)M'_{k,-m}(z_o) - M_{k,-m}(z_o)M'_{k,m}(z_o)$ is the Wronskian. From equation (104), the approximate density ratio is

$$\frac{\tilde{\rho}}{\rho_o} = \frac{1}{z_o c}\,\frac{EM_{k,m}(z) + FM_{k,-m}(z)}{EM'_{k,m}(z) + FM'_{k,-m}(z)} \tag{117}$$

which is plotted in figure 14. It differs very little from the exact curve. From equations (108a) and (108b), the approximate Mach number is

$$\tilde{M} = 1 - \frac{\left[EM_{k,m}(z) + FM_{k,-m}(z)\right]\left[EM''_{k,m}(z) + FM''_{k,-m}(z)\right]}{EM'_{k,m}(z) + FM'_{k,-m}(z)} \tag{118}$$

which is shown to compare reasonably well with the exact curve in figure 15.

The values of $\frac{dp}{d\rho} = \left(\tilde{a}*\right)^2$ and p against σ can be calculated from

$$\left(\tilde{a}*\right)^2 = \frac{\tilde{q}^2}{\tilde{M}^2}$$

$$= \frac{\left[EM'_{k,m}(z) + FM'_{k,-m}(z)\right]^2\left[EM''_{k,m}(z) + FM''_{k,-m}(z)\right]^{-1}}{\left[EM_{k,m}(z) + FM_{k,-m}(z)\right]\left\{\left[EM'_{k,m}(z) + FM'_{k,-m}(z)\right] - \left[EM_{k,m}(z) + FM_{k,-m}(z)\right]\right\}} \tag{119}$$

and

$$\tilde{p} = \int \rho_o \tilde{q}^2 \, d\sigma$$

$$= \frac{\rho_o}{cz_o}\int \frac{dz}{\left[EM_{k,m}(z) + FM_{k,-m}(z)\right]^2} \tag{120}$$

Since the hypothetical gas law differs so little from the exact value, it seems justifiable to use the exact gas law to replace the hypothetical gas law if necessary, particularly in the neighborhood of the sonic velocity.

To the author's knowledge, the only available high-order approximation is Loewners' approximation (reference 73). It is the basis of Carrier and Ehlers' investigation on channel flow (reference 32). For comparison, his approximation in gas behavior is given in figures 14 and 15. His approximation is correct at the sonic velocity to the second derivative of the $K - \sigma$ curve.

For the present second approximation $b = c$ or $m = -\frac{1}{2}$ is chosen. When $2m$ is an integer only one of the series solutions is valid, namely, $M_{k,-m}(z)$. If the solution $M_{k,-m}(z)$ is retained then the second independent solution can be obtained by using the limiting value of $M_{k,-m}(z)$ as $m \longrightarrow -\frac{1}{2}$. It is quite similar to the development of the preceding section. No further details are necessary.

For the case $2b = c$, $k = 0$. Similar solutions can be obtained with slightly different boundary conditions.

6 - TRANSFORMATION BETWEEN HODOGRAPH PLANE AND PHYSICAL PLANE

With the introduction of $d\sigma = -\frac{\rho_o}{\rho}\frac{dq}{q}$ the differential equations for ψ and ϕ from equations (10) and (16) become

$$\frac{\partial \phi}{\partial \sigma} = K(\sigma)\frac{\partial \psi}{\partial \theta} \tag{121a}$$

$$\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial \theta} \tag{121b}$$

where $K(\sigma) = \left(\frac{\rho_o}{\rho}\right)^2\left(1 - M^2\right)$. As shown before, the differential equation in ψ is

$$\frac{\partial^2 \psi}{\partial \sigma^2} + K(\sigma)\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{122}$$

with the exact value of $K(\sigma)$ approximated by

$$\tilde{K}_3(\sigma) = \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}$$

Introducing $\psi(\sigma,\theta) = \psi_1(\sigma)\psi_2(\theta)$,

$$\frac{d^2\psi_1}{d\sigma^2} - n^2\lambda^2\tilde{K}(\sigma)\psi_1 = 0 \tag{123}$$

where n is an integer and λ is positive and real; λ is introduced here in order to make $\psi_2(n\lambda\theta) = \psi_2(n\lambda\theta + 2\pi)$ periodic. For simplicity $\lambda = 1$ is chosen in the later treatment. With the introduction of $z = \frac{2n}{c^2}\sqrt{ab}(1 + c\sigma)$ (not to be confused with Z to be introduced presently), the general solution of equation (122) has been found to be

$$\psi(\sigma,\theta) = A_o\left(\sigma + \sigma_o\right)\left(\theta + \theta_o\right) + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z)\cos n\theta +$$

$$\sum_{n=1}^{\infty} B_n W_{k,m}^{(n)}(z)\sin n\theta \tag{124}$$

where A_o, A_n, B_n, σ_o, and θ_o are to be evaluated from the given boundary conditions for a particular problem at hand, or from the solution of the corresponding incompressible flow. The superscript (n) is applied to the Whittaker's function in order to show its relation to n. Suppose that all these constants are known and the right-hand side of equation (124) is assumed convergent and represents $\psi(\sigma,\theta)$; it remains to transform the results obtained in the hodograph plane to the physical plane so that the problem is solved in the physical plane. The procedure of carrying out such a transformation is given here.

First introduce q and θ, the inclination of the velocity vector, as

$$u = q\cos\theta$$

$$v = q\sin\theta$$

where $u = \frac{\partial\phi}{\partial X} = \frac{\rho_o}{\rho}\frac{\partial\psi}{\partial Y}$ is the local velocity component along the X-axis and $v = \frac{\partial\phi}{\partial Y} = -\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial X}$ is the local velocity component along the Y-axis. Then the total differentials of ϕ, ψ are

$$d\phi = \frac{\partial \phi}{\partial X}\,dX + \frac{\partial \phi}{\partial Y}\,dY$$

$$= q(\cos\theta\,dX + \sin\theta\,dY) \tag{125a}$$

$$d\psi = \frac{\partial \psi}{\partial X}\,dX + \frac{\partial \psi}{\partial Y}\,dY$$

$$= \frac{\rho}{\rho_o}\,q(-\sin\theta\,dX + \cos\theta\,dY) \tag{125b}$$

Or, in short,

$$dZ = \frac{e^{i\theta}}{q}\left(d\phi + i\,\frac{\rho_o}{\rho}\,d\psi\right) \tag{125c}$$

where $Z = X + iY$ is a complex variable in the physical plane. With σ, θ as independent variables in the physical plane,

$$d\phi = \frac{\partial \phi}{\partial \sigma}\,d\sigma + \frac{\partial \phi}{\partial \theta}\,d\theta \tag{126a}$$

$$d\psi = \frac{\partial \psi}{\partial \sigma}\,d\sigma + \frac{\partial \psi}{\partial \theta}\,d\theta \tag{126b}$$

The derivatives of ϕ can be eliminated entirely if the relations given in equations (121a) and (121b) are introduced in equation (126a)

$$d\phi = K(\sigma)\,\frac{\partial \psi}{\partial \theta}\,d\sigma - \frac{\partial \psi}{\partial \sigma}\,d\theta \tag{127}$$

Substituting equations (126b) and (127) into equation (125c), equation (125c) can be rewritten as the exact differential

$$dZ = \frac{\partial Z}{\partial \sigma} d\sigma + \frac{\partial Z}{\partial \theta} d\theta$$

$$= \frac{e^{i\theta}}{q} \left\{ \left[K(\sigma) \frac{\partial \psi}{\partial \theta} + i \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial \sigma} \right] d\sigma + \left(- \frac{\partial \psi}{\partial \sigma} + i \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial \theta} \right) d\theta \right\} \tag{128a}$$

or

$$\frac{\partial Z}{\partial \sigma} = \frac{e^{i\theta}}{q} \left[K(\sigma) \frac{\partial \psi}{\partial \theta} + i \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial \sigma} \right] \tag{128b}$$

$$\frac{\partial Z}{\partial \theta} = \frac{e^{i\theta}}{q} \left(- \frac{\partial \psi}{\partial \sigma} + i \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial \theta} \right) \tag{128c}$$

The above relations are conformal and uniquely exist as long as the Jacobian determinate

$$\frac{\partial(X,Y)}{\partial(\sigma,\theta)} \neq 0 \quad \text{or} \quad \infty \tag{129}$$

In the case of $\frac{\partial(X,Y)}{\partial(\sigma,\theta)} = 0$ at (X,Y) some singularity may occur in the physical plane. If $\frac{\partial(X,Y)}{\partial(\sigma,\theta)} = \infty$ some singularity may occur in the hodograph plane as shown in reference 48.

Before carrying out the calculation, a few symbols are introduced to make the presentation a little clearer. Let

$$\left.\begin{aligned}
\psi^{(0)} &= (\sigma + \sigma_o)(\theta + \theta_o) \\
\psi_c^{(n)} &= W_{k,m}^{(n)}(z) \cos n\theta = W^{(n)} \cos n\theta \\
\psi_s^{(n)} &= W_{k,m}^{(n)}(z) \sin n\theta = W^{(n)} \sin n\theta
\end{aligned}\right\} \tag{130}$$

Also represent $Z = Z^{(o)}$ corresponding to $\psi^{(o)}$ and $Z = Z^{(n)}$ to $\psi^{(n)}(z)$. For future use, rewrite equation (106) as

$$\frac{dq}{d\sigma} = -q \frac{\rho_o}{\rho} \tag{131a}$$

$$\frac{1}{\rho}\frac{d\rho}{d\sigma} = \frac{\rho_o}{\rho} M^2 \tag{131b}$$

and

$$\frac{d}{d\sigma}\left(\frac{\rho_o}{\rho q}\right) = \frac{K}{q} \tag{131c}$$

To carry out the evaluation of Z in terms of σ and θ is rather too long for the space available in this report. The basic method of calculation is rather simple. Since in equation (128a) $dZ = \frac{\partial Z}{\partial \sigma} d\sigma + \frac{\partial Z}{\partial \theta} d\theta$ is an exact differential, $\frac{\partial Z}{\partial \theta}$ $\left(\text{or } \frac{\partial Z}{\partial \sigma}\right)$ can be integrated to obtain Z except for an unknown function $F(\sigma)$ (or $F(\theta)$) to be determined. The unknown function $F(\sigma)$ (or $F(\theta)$) can be evaluated from the remaining relation $\frac{\partial Z}{\partial \sigma}$ $\left(\text{or } \frac{\partial Z}{\partial \theta}\right)$. Of course, in the operation, repeated use has been made of the differential equation $\frac{d^2\psi_2(\sigma)}{d\sigma^2} - n^2K\psi_2(\sigma) = 0$ or Whittaker's equation. For further details the following section may be consulted.

<u>n = 0</u>.- In this case,

$$Z^{(o)} = X^{(o)} + iY^{(o)}$$

$$= \frac{e^{i\theta}}{q}\left\{\left[\frac{\rho_o}{\rho}\left(\sigma + \sigma_o\right) - 1\right] - i\left(\theta + \theta_o\right)\right\} + Z_o^{(o)} \tag{132}$$

where $Z_o^{(o)} = X_o^{(o)} + iY_o^{(o)}$ is the constant of integration.

n = 1.- For n = 1, the function $F^{(1)}(\sigma)$ is an integral as

$$F^{(1)}(\sigma) = \frac{1}{2} \int \left[\frac{\rho_o}{\rho q} \frac{dW^{(1)}}{d\sigma} - \frac{K}{q} W^{(1)}\right] dq \tag{133}$$

The term $Z_c^{(1)}$ can be expressed by

$$Z_c^{(1)} - Z_{c_o}^{(1)} = \frac{ie^{2i\theta}}{4q}\left[\frac{dW^{(1)}}{d\sigma} + \frac{\rho_o}{\rho} W^{(1)}\right] - \frac{\theta}{2q}\left[\frac{dW^{(1)}}{d\sigma} - \frac{\rho_o}{\rho} W^{(1)}\right] +$$

$$\frac{i}{2} \int \frac{1}{q}\left[\frac{\rho_o}{\rho} \frac{dW^{(1)}}{d\sigma} - KW^{(1)}\right] d\sigma \tag{134a}$$

and

$$Z_s^{(1)} - Z_{s_o}^{(1)} = \frac{e^{2i\theta}}{4q}\left[\frac{dW^{(1)}}{d\sigma} + \frac{\rho_o}{\rho} W^{(1)}\right] - \frac{i\theta}{2q}\left[\frac{dW^{(1)}}{d\sigma} - \frac{\rho_o}{\rho} W^{(1)}\right] +$$

$$\frac{1}{2} \int \frac{1}{q}\left[\frac{\rho_o}{\rho} \frac{dW^{(1)}}{d\sigma} - KW^{(1)}\right] d\sigma \tag{134b}$$

where $Z_{c_o}^{(1)}$ and $Z_{s_o}^{(1)}$ are the integration constants.

n ≠ 1 or 0.- For the cases $n \neq 1$ or 0, the function $F^{(n)}(\sigma)$ is found to be a constant

$$Z_c^{(n)} - Z_{c_o}^{(n)} = \frac{ie^{i(n+1)\theta}}{2(n + 1)q}\left[\frac{dW^{(n)}}{d\sigma} + \frac{n\rho_o}{\rho} W^{(n)}\right] -$$

$$\frac{ie^{-i(n-1)\theta}}{2(n - 1)q}\left[\frac{dW^{(n)}}{d\sigma} - \frac{n\rho_o}{\rho} W^{(n)}\right] \tag{135a}$$

and

$$Z_s^{(n)} - Z_{s_o}^{(n)} = \frac{e^{i(n+1)\theta}}{2(n+1)q}\left[\frac{dW^{(n)}}{d\sigma} + \frac{n\rho_o}{\rho} W^{(n)}\right] +$$

$$\frac{e^{-i(n-1)\theta}}{2(n-1)q}\left[\frac{dW^{(n)}}{d\sigma} - \frac{n\rho_o}{\rho} W^{(n)}\right] \qquad (135b)$$

From the earlier definition,

$$Z = A_o Z^{(o)} + A_1 Z_c^{(1)} + B_1 Z_s^{(1)} + \sum_{n=2}^{\infty}\left[A_n Z_c^{(n)} + B_n Z_s^{(n)}\right] \qquad (136a)$$

and the constant term is

$$Z_o = A_o Z_o^{(o)} + A_1 Z_{c_o}^{(1)} + B_1 Z_{s_o}^{(1)} + \sum_{n=2}^{\infty}\left[A_n Z_{c_o}^{(n)} + B_n Z_{s_o}^{(n)}\right] \qquad (136b)$$

Substituting equations (132), (134a), (134b), (135a), and (135b) into the above equation,

$$Z - Z_o = \frac{A_o e^{i\theta}}{q}\left[\frac{\rho_o}{\rho}\left(\sigma + \sigma_o\right) - 1 - i\left(\theta + \theta_o\right)\right] + \frac{\left(iA_1 + B_1\right)e^{2i\theta}}{4q}\left[\frac{dW^{(1)}}{d\sigma} +\right.$$

$$\left.\frac{\rho_o}{\rho} W^{(1)}\right] - \frac{\left(A_1 + iB_1\right)\theta}{2q}\left[\frac{dW^{(1)}}{d\sigma} - \frac{\rho_o}{\rho} W^{(1)}\right] +$$

$$\frac{iA_1 + B_1}{2}\int\frac{1}{q}\left[\frac{\rho_o}{\rho}\frac{dW^{(1)}}{d\sigma} - KW^{(1)}\right]d\sigma +$$

$$\sum_{n=2}^{\infty}\left\{\frac{\left(iA_n + B_n\right)e^{i(n+1)\theta}}{2(n+1)q}\left[\frac{dW^{(n)}}{d\sigma} + \frac{n\rho_o}{\rho} W^{(n)}\right] +\right.$$

$$\left.\frac{\left(-iA_n + B_n\right)e^{-i(n-1)\theta}}{2(n-1)q}\left[\frac{dW^{(n)}}{d\sigma} - \frac{n\rho_o}{\rho} W^{(n)}\right]\right\} \qquad (137a)$$

In more detail, the expression is

$$Z - Z_o = \frac{A_o e^{i\theta}}{q}\left[\frac{\rho_o}{\rho}\left(\sigma + \sigma_o\right) - 1 - i\left(\theta + \theta_o\right)\right] + \frac{\left(iA_1 + B_1\right)e^{2i\theta}}{4q}\left[\frac{dW_{k,m}^{(1)}(z)}{d\sigma} + \right.$$

$$\left.\frac{\rho_o}{\rho}W_{k,m}^{(1)}(z)\right] - \frac{\left(A_1 + iB_1\right)\theta}{2q}\left[\frac{dW_{k,m}^{(1)}(z)}{d\sigma} - \frac{\rho_o}{\rho}W_{k,m}^{(1)}(z)\right] +$$

$$\frac{iA_1 + B_1}{2}\int \frac{1}{q}\left[\frac{\rho_o}{\rho}\frac{dW_{k,m}^{(1)}(z)}{d\sigma} - K(\sigma)W_{k,m}^{(1)}(z)\right] d\sigma +$$

$$\sum_{n=2}^{\infty}\left\{\frac{\left(iA_n + B_n\right)e^{i(n+1)\theta}}{2(n + 1)q}\left[\frac{dW_{k,m}^{(n)}(z)}{d\sigma} + \frac{n\rho_o}{\rho}W_{k,m}^{(n)}(z)\right] + \right.$$

$$\left.\frac{\left(-iA_n + B_n\right)e^{-i(n-1)\theta}}{2(n - 1)q}\left[\frac{dW_{k,m}^{(n)}(z)}{d\sigma} - \frac{n\rho_o}{\rho}W_{k,m}^{(n)}(z)\right]\right\} \tag{137b}$$

where $z = \frac{2n}{c^2}\sqrt{ab}(1 + c\sigma)$.

The above result is also true for the cases of the second approximation $b = c$ $\left(m = -\frac{1}{2}\right)$, or of the approximation $b = 2c$ $(k = 0)$.

7 - FLOW OF COMPRESSIBLE FLUID THROUGH AN APERTURE OF A TWO-DIMENSIONAL INCLINED-WALLED, STRAIGHT-EDGED NOZZLE

To apply a critical test to the present investigation it seems reasonable to compare the present approximation with a well-known flow pattern which has been studied by early explorers using the exact method. For instance, Chaplygin gave an application of his investigation to the efflux of a gas from an infinite vessel. Later Lighthill, following his approach, repeated the example. Both considered the particular case with the walls normal to the jet and both brought the boundary conditions from the incompressible flow and treated the maximum jet velocity up to the sound velocity.

The present application gives a more general type of such flow - flow through an aperture of an inclined-walled, straight-edged nozzle. Besides, the problem is treated directly as a boundary-value problem.

The question of the maximum velocity in the jet depends on the value of the ratio of the pressure p_∞ surrounding the jet to the stagnation pressure p_0 in the vessel. As long as $\frac{p_\infty}{p_0} \geqq \left(\frac{\gamma + 1}{2}\right)^{-\frac{\gamma}{\gamma-1}}$, the maximum velocity can never exceed the sound velocity. The boundary value of the stream function ψ is clearly defined; therefore it is a direct boundary-value problem. For the case $\frac{p_\infty}{p_0} < \left(\frac{\gamma + 1}{2}\right)^{-\frac{\gamma}{\gamma-1}}$ the velocity of the jet will be supersonic and this problem is as yet unsolved.

The approximate differential equation used is (given in equation (84a))

$$\frac{\partial^2\psi}{\partial\sigma^2} + \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{138}$$

and its general solution has been found to be

$$\begin{aligned}\psi(\sigma,\theta) &= \left(c_1\sigma + c_2\right)\left(c_3\theta + c_4\right) + \sum_{n=1}^{\infty} \left(\bar{A}_n \cos n\lambda\theta + \bar{B}_n \sin n\lambda\theta\right)W_{k,m}^{(n)}(z) \\ &= \left(c_1\sigma + c_2\right)\left(c_3\theta + c_4\right) + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin\left(n\lambda\theta + \alpha_n\right)\end{aligned} \tag{139}$$

where c_1, c_2, c_3, c_4, A_n, B_n, and λ are constants; $A_n \sin \alpha_n = \overline{A}_n$ and $A_n \cos \alpha_n = \overline{B}_n$ are obvious; $z = \frac{2n\lambda}{c^2}\sqrt{ab}(1 + c\sigma) \geqq 0$,

$k = \frac{-n\lambda(c - 2b)}{2c^2}\sqrt{\frac{a}{b}} < 0$, and $m = \sqrt{\frac{1}{4} - \frac{(n\lambda)^2 a(c - b)}{c^4}}$; and n is a positive integer. It represents the required solution if the series converges.

The constants c_1, c_2, c_3, c_4, A_n, α_n, and λ are determined from the boundary conditions which are shown as follows.

θ_o arbitrary.- The flow in the physical plane is shown in figure 16(a) and in the hodograph planes in figures 16(b) and 16(c). Since it is known that the discharge Q from the aperture is finite, the stream function ψ is bounded. Moreover, the flow is symmetrical with respect to the center line. If the rate of total discharge is introduced as Q, then for inside the vessel there can be written

$$\left.\begin{aligned} \psi &= -\frac{Q}{2} \quad \text{at} \quad \theta = \theta_o \\ \psi &= 0 \quad \text{at} \quad \theta = 0 \\ \psi &= \frac{Q}{2} \quad \text{at} \quad \theta = -\theta_o \end{aligned}\right\} \tag{140}$$

Thus it is obvious that ψ is an odd function of θ. Corresponding to a constant p_∞, q and σ on the outer surface of the jet are constant from consideration of Bernoulli's equation. Thus $\psi = -\frac{Q}{2}$ at

$$\left.\begin{aligned} &\sigma = \sigma_o = \text{Constant} \\ \text{and} \qquad & \\ &0 < \theta \leqq \theta_o \end{aligned}\right\} \tag{141}$$

First of all, since $\psi = 0$ at $\theta = 0$,

$$\left.\begin{aligned} \alpha_n &= 0 \\ c_4 &= 0 \end{aligned}\right\} \tag{142}$$

Thus equation (139) becomes

$$\psi(\sigma,\theta) = \left(c_1\sigma + c_2\right) c_3\theta + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin n\lambda\theta \tag{143}$$

which is an odd function of θ as required. Next, the stream function is defined in the interval $-\theta_0 \leqq \theta \leqq \theta_0$ or the period $2\theta_0$. Thus (when n is an integer) $\sin n\lambda\left(\theta + 2\theta_0\right) = \sin n\lambda\theta$ or $2n\lambda\theta_0 = 2n\pi$.

$$\lambda = \frac{\pi}{\theta_0} \tag{144}$$

Substituting equation (144) into equation (143),

$$\psi(\sigma,\theta) = \left(c_1\sigma + c_2\right)c_3\theta + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin \frac{n\pi}{\theta_0}\theta \tag{145}$$

Now, since ψ is bounded everywhere, particularly when $\sigma \longrightarrow \infty$ or $M \longrightarrow 0$, c_1 must be equal to 0. Thus equation (145) yields

$$\psi(\sigma,\theta) = A_0\theta + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin \frac{n\pi}{\theta_0}\theta \tag{146}$$

At $\theta = \theta_o$, $\psi = -\frac{Q}{2}$,

$$\left.\begin{aligned} -\frac{Q}{2} &= A_o\theta_o \\ A_o &= -\frac{Q}{2\theta_o} \end{aligned}\right\} \tag{147}$$

Thus

$$\psi(\sigma,\theta) = -\frac{Q\theta}{2\theta_o} + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}(z) \sin \frac{n\pi}{\theta_o}\theta \tag{148}$$

Finally the constants A_n can be determined from the boundary condition $\psi = -\frac{Q}{2}$ at $\sigma = \sigma_o$ or $z = z_o = \frac{2n\pi}{\theta_o c^2}\sqrt{ab}\left(1 + c\sigma_o\right)$ for $0 < \theta \leqq \theta_o$.

$$\psi = -\frac{Q}{2} = -\frac{Q}{2}\frac{\theta}{\theta_o} + \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}\left(z_o\right) \sin \frac{n\pi\theta}{\theta_o} \tag{149}$$

or

$$-\frac{Q}{2}\left(1 - \frac{\theta}{\theta_o}\right) = \sum_{n=1}^{\infty} A_n W_{k,m}^{(n)}\left(z_o\right) \sin \frac{n\pi\theta}{\theta_o}$$

Now the left-hand side of the equation is a known function θ/θ_o and the right-hand side is the Fourier series of $\frac{Q}{2}\left(1 - \frac{\theta}{\theta_o}\right)$ if the series converges. It is easy to determine the coefficient as

$$\begin{aligned} A_n W_{k,m}^{(n)}\left(z_o\right) &= -\int_{-1}^{1} \frac{Q}{2}\left(1 - \frac{\theta}{\theta_o}\right) \sin\left(n\pi\frac{\theta}{\theta_o}\right) d\left(\frac{\theta}{\theta_o}\right) \\ &= -\frac{Q}{n\pi} \end{aligned} \tag{150}$$

or

$$A_n = -\frac{Q}{n\pi}\frac{1}{W_{k,m}^{(n)}\left(z_o\right)}$$

Substituting into equation (148),

$$\psi(\sigma,\theta) = -\left[\frac{Q}{2}\frac{\theta}{\theta_o} + \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{W_{k,m}^{(n)}(z)}{nW_{k,m}^{(n)}\ z_o}\sin\left(n\pi\frac{\theta}{\theta_o}\right)\right] \tag{151}$$

First, it is necessary to show that equation (151) is valid for an incompressible flow which can be considered as the limiting case of $a^* \longrightarrow \infty$. From the definition,

$$\sigma = \int_q^{a^*}\frac{\rho}{\rho_o}\frac{dq}{q}$$

there results, for incompressible flow,

$$\sigma_i = \lim_{a^*\longrightarrow\infty}\sigma = \lim_{a^*\longrightarrow\infty}\log_e\frac{a^*}{q} \tag{152}$$

and, from the definition of z for very large values of σ, $\left(\text{since}\ \frac{\sqrt{ab}}{c} = 1\right)$

$$z = \frac{2n\lambda}{c}(1 + c\sigma)$$

$$= 2n\lambda\sigma\left[1 + O\left(\sigma^{-1}\right)\right]$$

From equation (87) the asymptotic expansion of $W_{k,m}^{(n)}(z)$ for large values of σ is

$$W_{k,m}^{(n)}\ (z) = e^{-\frac{z}{2}}\ z^k\left[1 + O\left(\frac{1}{\sigma}\right)\right]$$

$$= e^{-n\lambda\sigma}(2n\lambda\sigma)^k\left[1 + O\left(\frac{1}{\sigma}\right)\right]$$

Then the ratio of the two Whittaker functions in equation (151) at very large values of σ is

$$\frac{W_{k,m}^{(n)}\ (z)}{W_{k,m}^{(n)}\ \left(z_o\right)} = e^{-n\lambda(\sigma-\sigma_o)}\left(\frac{\sigma}{\sigma_o}\right)^k\left[1 + O\left(\frac{1}{\sigma}\right)\right] \tag{153}$$

Now, for an incompressible flow with equation (152),

$$\sigma_i - \sigma_{o_i} = \lim_{a^*=a_o^*\longrightarrow\infty}\left(\log_e \frac{a^*}{q} - \log_e \frac{a_o^*}{q_o}\right)$$

$$= \lim_{a^*=a_o^*\longrightarrow\infty}\left(\log_e \frac{a^*}{a_o^*} - \log_e \frac{q}{q_o}\right)$$

$$= -\log_e \frac{q}{q_o}$$

and

$$\frac{\sigma_i}{\sigma_{o_i}} = \lim_{a^*=a_o^*\longrightarrow\infty} \frac{\sigma}{\sigma_o} = \lim_{a^*=a_o^*\longrightarrow\infty} \frac{\log_e a^* - \log_e q}{\log_e a_o^* - \log_e q_o} = 1$$

Introducing the above results into equation (153),

$$\left[\frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_0)}\right]_i = \lim_{a^*=a_0^*\longrightarrow\infty} \frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_0)} = \left(\frac{q}{q_0}\right)^{n\lambda} = \left(\frac{q}{q_0}\right)^{\frac{n\pi}{\theta_0}}$$

Thus the stream function for the incompressible flow is

$$\psi_i = -\left[\frac{Q}{2}\frac{\theta}{\theta_0} + \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{q}{q_0}\right)^{\frac{n\pi}{\theta_0}} \sin\left(n\pi\frac{\theta}{\theta_0}\right)\right] \tag{154}$$

which reduces to Chaplygin's case if θ_0 is set equal to $\pi/2$.

The convergence of the series is easily proved, because $\frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_0)} \leqq 1$ for $z \geqq z_0$ and $W_{k,m}^{(n)}(z) > 0$ is a monotonically decreasing function of z in the subsonic range (see appendix B). Compare the nth term of equation (151) with $\frac{1}{n}\sin n\pi\frac{\theta}{\theta_0}$ which is a term of a convergent series, and each corresponding term is smaller. Therefore, equation (151) is convergent.

The main interest of this problem has been the minimum width of the jet which occurs at $X = \infty$. Therefore the solution must be transformed back to the physical plane.

With the differential relation from equations (1a) and (1b), the following equations can be derived:

$$\left.\begin{aligned}\frac{\partial\phi}{\partial\sigma} &= K\frac{\partial\psi}{\partial\theta}\\[2ex] \frac{\partial\phi}{\partial\theta} &= -\frac{\partial\psi}{\partial\sigma}\end{aligned}\right\} \tag{155}$$

and, rewriting equation (128a),

$$dZ = dX + i\ dY$$

$$= \frac{e^{i\theta}}{q}\left[\left(K \frac{\partial\psi}{\partial\theta} + i \frac{\rho_o}{\rho} \frac{\partial\psi}{\partial\sigma}\right) d\sigma + \left(- \frac{\partial\psi}{\partial\sigma} + i \frac{\rho_o}{\rho} \frac{\partial\psi}{\partial\theta}\right) d\theta\right] \tag{156}$$

Differentiating equation (151) with respect to σ and θ and remembering $z_\sigma = \frac{dz}{d\sigma} = \frac{2n\pi}{\theta_o c}\sqrt{ab} = \frac{2n\pi}{\theta_o}$ $\left(\text{since } \frac{\sqrt{ab}}{c} = 1\right)$,

$$- \frac{\partial\psi}{\partial\sigma} = - \frac{\partial\psi}{\partial z} \frac{dz}{d\sigma}$$

$$= \frac{Q}{\pi} \sum_{n=1}^{\infty} \frac{2\pi}{c\theta_o} \frac{W_{k,m}'^{(n)}(z)}{W_{k,m}^{(n)}(z_o)} \sin\left(n\pi \frac{\theta}{\theta_o}\right)$$

$$= \frac{2Q}{\theta_o} \sum_{n=1}^{\infty} \frac{W_{k,m}'^{(n)}(z)}{W_{k,m}^{(n)}(z_o)} \sin\left(\frac{n\pi\theta}{\theta_o}\right) \tag{157a}$$

$$- \frac{\partial\psi}{\partial\theta} = \frac{Q}{2\theta_o} + \frac{Q}{\theta_o} \sum_{n=1}^{\infty} \frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_o)} \cos \frac{n\pi\theta}{\theta_o} \tag{157b}$$

Substituting the above equations into equation (156),

$$-dZ = \frac{e^{i\theta}}{q}\left\{K\,\frac{Q}{\theta_o}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_o)}\cos\frac{n\pi\theta}{\theta_o}\right] + \right.$$

$$\left. i\,\frac{2\rho_o Q}{\rho\theta_o}\sum_{n=1}^{\infty}\frac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z_o)}\sin\frac{n\pi\theta}{\theta_o}\right\}d\sigma +$$

$$\frac{e^{i\theta}}{q}\left\{-\frac{2Q}{\theta_o}\sum_{n=1}^{\infty}\frac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z_o)}\sin\frac{n\pi\theta}{\theta_o} + \right.$$

$$\left.\frac{i\rho_o Q}{\rho\theta_o}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_o)}\cos\frac{n\pi\theta}{\theta_o}\right]\right\}d\theta \qquad (158)$$

which is the conformal relation of any elementary length in the hodograph plane to the corresponding element in the physical plane as long as the Jacobian determinate $\frac{\partial(X,Y)}{\partial(\sigma,\theta)} \neq 0$. This condition is automatically fulfilled in the subsonic region up to sonic velocity as has been shown by Tsien (reference 45) and Craggs (reference 48). Since dZ is an exact differential, there can be introduced,

$$-\frac{\partial Z}{\partial\sigma} = \frac{e^{i\theta}}{q}\left\{\frac{QK}{\theta_o}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{W_{k,m}^{(n)}(z)}{W_{k,m}^{(n)}(z_o)}\cos\frac{n\pi\theta}{\theta_o}\right] + \right.$$

$$\left.\frac{i2\rho_o Q}{\rho\theta_o}\sum_{n=1}^{\infty}\frac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z_o)}\sin\frac{n\pi\theta}{\theta_o}\right\} \qquad (159a)$$

and

$$-\frac{\partial Z}{\partial \theta} = \frac{e^{i\theta}}{q}\left\{-\frac{2Q}{\theta_o}\sum_{n=1}^{\infty}\frac{W'^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\sin\frac{n\pi\theta}{\theta_o} + \frac{i\rho_o Q}{\rho\theta_o}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\cos\frac{n\pi\theta}{\theta_o}\right]\right\} \tag{159b}$$

The value of Z can be determined by integrating $\frac{\partial Z}{\partial \theta}$, leaving a function of σ to be determined, and, consequently,

$$-Z = \frac{Qe^{i\theta}}{\theta_o q}\sum_{n=1}^{\infty}\frac{2W'^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\left[\frac{\frac{n\pi}{\theta_o}\cos\frac{n\pi}{\theta_o}\theta - i\sin\frac{n\pi}{\theta_o}\theta}{\left(\frac{n\pi}{\theta_o}\right)^2 - 1}\right] + \frac{Q\rho_o e^{i\theta}}{\theta_o\rho q}\sum_{n=1}^{\infty}\frac{W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\left[\frac{i\frac{n\pi}{\theta_o}\sin\frac{n\pi}{\theta_o}\theta - \cos\frac{n\pi}{\theta_o}\theta}{\left(\frac{n\pi}{\theta_o}\right)^2 - 1}\right] + \frac{Q\rho_o e^{i\theta}}{2\theta_o\rho q} + F(\sigma) \tag{160}$$

where $F(\sigma)$ is to be determined. This can be done by differentiating equation (160)

$$-\frac{\partial Z}{\partial \sigma} = \frac{Qe^{i\theta}}{\theta_o q}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi\theta}{\theta_o}}{\left[\left(\frac{n\pi}{\theta_o}\right)^2 - 1\right]W^{(n)}_{k,m}(z_o)}\left[z_\sigma^{\,2}W''^{(n)}_{k,m}(z) - KW^{(n)}_{k,m}(z)\right] + \frac{iQe^{i\theta}}{\theta_o q}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\theta}{\theta_o}}{\left[\left(\frac{n\pi}{\theta_o}\right)^2 - 1\right]W^{(n)}_{k,m}(z_o)}\left\{-2z_\sigma W''^{(n)}_{k,m}(z) - 2\left[\left(\frac{n\pi}{\theta_o}\right)^2 - 1\right]\frac{\rho_o}{\rho}W'^{(n)}_{k,m}(z) + \frac{n\pi}{\theta_o}KW^{(n)}_{k,m}(z)\right\} + \frac{Qe^{i\theta}}{2\theta_o q}K + F'(\sigma) \tag{161}$$

where $z_\sigma = \dfrac{2n\pi}{\theta_o}$, and equations (131a) and (131b) are used. Equating equations (159a) and (161),

$$F'(\sigma) = \frac{Qe^{i\theta}}{\theta_o q}\sum_{n=1}^{\infty}\frac{\cos\dfrac{n\pi\theta}{\theta_o}}{\left[\left(\dfrac{n\pi}{\theta_o}\right)^2 - 1\right]}\left[z_\sigma^2 W''^{(n)}_{k,m}(z) - \frac{n^2\pi^2}{\theta_o^2}KW^{(n)}_{k,m}(z)\right]\frac{1}{W^{(n)}_{k,m}(z_o)} -$$

$$\frac{iQe^{i\theta}}{\theta_o q}\sum_{n=1}^{\infty}\frac{\dfrac{n\pi}{\theta_o}\sin\dfrac{n\pi\theta}{\theta_o}}{\left[\left(\dfrac{n\pi}{\theta_o}\right)^2 - 1\right]W^{(n)}_{k,m}(z_o)}\left[z_\sigma^2 W''^{(n)}_{k,m}(z) - \right.$$

$$\left.\frac{n^2\pi^2}{\theta_o^2}KW^{(n)}_{k,m}(z)\right] = 0 \tag{162}$$

The expressions in the brackets of the right-hand side of equation (162) are identically equal to the differential equation, equation (78) if $\nu^2 = \dfrac{n^2\pi^2}{\theta_o^2}$ and consequently are zero as shown.

Thus, choose

$$F(\sigma) = \text{Constant} = Z_\sigma \tag{163}$$

Now equation (160) can be rewritten in a more suitable form

$$Z = \frac{-\rho_o Q}{2\rho\theta_o q}\left\{e^{i\theta} + \sum_{n=1}^{\infty}\left[\frac{2\dfrac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) + W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{e^{i\left(\frac{n\pi}{\theta_o}+1\right)\theta}}{\dfrac{n\pi}{\theta_o}+1} + \right.\right.$$

$$\left.\left.\frac{2\dfrac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) - W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{e^{-i\left(\frac{n\pi}{\theta_o}-1\right)\theta}}{\dfrac{n\pi}{\theta_o}-1}\right]\right\} \tag{164a}$$

Specifically,

$$X = \frac{-\rho_o Q}{2\rho\theta_o q}\left\{\cos\theta + \sum_{n=1}^{\infty}\left[\frac{2\frac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) + W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{\cos\left(\frac{n\pi}{\theta_o}+1\right)\theta}{\frac{n\pi}{\theta_o}+1} + \frac{2\frac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) - W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{\cos\left(\frac{n\pi}{\theta_o}-1\right)\theta}{\frac{n\pi}{\theta_o}-1}\right]\right\} \tag{164b}$$

$$Y = \frac{-\rho_o Q}{2\rho\theta_o q}\left\{\sin\theta + \sum_{n=1}^{\infty}\left[\frac{2\frac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) + W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{\sin\left(\frac{n\pi}{\theta_o}+1\right)\theta}{\frac{n\pi}{\theta_o}+1} - \frac{2\frac{\rho}{\rho_o}W'^{(n)}_{k,m}(z) - W^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z_o)}\,\frac{\sin\left(\frac{n\pi}{\theta_o}-1\right)\theta}{\frac{n\pi}{\theta_o}-1}\right]\right\} \tag{164c}$$

The convergence of the above two series must be established. First, at $\sigma = \sigma_o$ or $z = z_o$, equation (164b) does not converge for $\theta = 0$, because the velocity vector of $q = q_\infty$ at $\theta = 0$ is located at $X = \infty$.

At very large values of n,

$$\left.\begin{aligned}
z &= \frac{2n\lambda}{c^2}\sqrt{ab}\left(1 + c\sigma_o\right) = \frac{2n\lambda}{c}\left(1 + c\sigma_o\right) \qquad \left(\text{since } \frac{\sqrt{ab}}{c} = 1\right)\\
k &= -n\lambda\,\frac{c - 2b}{2c^2}\sqrt{\frac{a}{b}} = -n\lambda\left(\frac{1}{2b} - \frac{1}{c}\right)\\
m &= \sqrt{\frac{1}{4} - \frac{n^2\lambda^2 a(c - b)}{c^4}} = \frac{in\lambda}{c^2}\sqrt{a(c - b)}\left[1 + O\left(\frac{1}{n}\right)\right] = \frac{in\lambda}{c}\left(\frac{a}{c} - 1\right)^{1/2}\left[1 + O\left(\frac{1}{n}\right)\right]\\
\overline{m} &= \frac{m}{z} = \frac{i}{2}\left(\frac{a}{c} - 1\right)^{1/2}\frac{1}{1 + c\sigma_o}\left[1 + O\left(\frac{1}{n}\right)\right]\\
\overline{k} &= \frac{k}{z} = -\left(\frac{c}{4b} - \frac{1}{2}\right)\frac{1}{1 + c\sigma_o}\left[1 + O\left(\frac{1}{n}\right)\right]
\end{aligned}\right\} \tag{165}$$

With the above notations the asymptotic expansion of $W_{k,m}^{(n)}(z)$ for large values of n can be introduced as follows:

$$W_{k,m}^{(n)}(z) = \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)\Gamma\left(k + \frac{1}{2} - m\right)\left[\frac{z^{1/6}}{3\left(\frac{1}{4} - |\overline{m}|\right)^2}\right]\left[ze^{4\overline{k}-1}\left(\frac{1}{4} + |\overline{m}|\right)\right]^m\left[1 + O\left(\frac{1}{n}\right)\right] \tag{166}$$

at sonic velocity ($\sigma = 0$). For the subsonic velocity ($\sigma > 0$), a similar expression should be given; the case $\sigma = 0$ is more critical. Furthermore from reference 61, page 352, example 3,

$$\frac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z)} = \frac{k}{z} - \frac{1}{2} - \frac{\left[m^2 - \left(k - \frac{1}{2}\right)^2\right]}{z}\frac{W_{k-1,m}^{(n)}(z)}{W_{k,m}^{(n)}(z)} \tag{167}$$

Now it is necessary to show that, for large values of n, the value of $\dfrac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z)}$ does not depend on n to the first order. Substituting equation (166) into equation (167),

$$\frac{W_{k,m}^{'(n)}(z)}{W_{k,m}^{(n)}(z)} = \overline{k} - \frac{1}{2} - \frac{\left[m^2 - \left(k - \frac{1}{2}\right)^2\right]}{z}\frac{\Gamma\left(k - \frac{1}{2} - m\right)}{\Gamma\left(k + \frac{1}{2} - m\right)}e^{-4\overline{m}}\left[1 + O\left(\frac{1}{n}\right)\right]$$

$$= \overline{k} - \frac{1}{2} + (\overline{m} + \overline{k})e^{-4\overline{m}}\left[1 + O\left(\frac{1}{n}\right)\right] \tag{168}$$

which is a constant independent of $1/n$ to the first order for large values of n. Therefore the coefficients involving Whittaker's function for large values of n can be written as

$$\frac{2\rho}{\rho_o}\frac{W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} + 1 \sim \frac{2\rho}{\rho_o}\left[\overline{k} - \frac{1}{2} + (\overline{m} + \overline{k})e^{-4\overline{m}}\right] + 1 \tag{169a}$$

$$\frac{2\rho}{\rho_o}\frac{W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} - 1 \sim \frac{2\rho}{\rho_o}\left[\bar{k} - \frac{1}{2} + (\bar{m} + \bar{k})e^{-4\bar{m}}\right] - 1 \tag{169b}$$

both of which are bounded. For the case $\theta_o = \frac{\pi}{2}, \frac{\pi}{3}, \ldots, \frac{\pi}{s}$ (s is an integer) the series in equation (164c) is convergent. This can be shown at the surface of the jet. The equation

$$Y = -\frac{s\rho_o Q}{2\pi\rho_\infty q_\infty}\left(\sin\theta + \sum_{n=1}^{\infty}\left\{\left[\frac{2\frac{\rho_\infty}{\rho_o}W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} + 1\right]\frac{\sin(sn+1)\theta}{sn+1} - \left[\frac{2\frac{\rho_\infty}{\rho_o}W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} - 1\right]\frac{\sin(sn-1)\theta}{sn-1}\right\}\right) \tag{170}$$

is dominated by $\frac{s\rho_o Q}{2\pi\rho_\infty q_\infty}\left(1 + E\sum_{n=1}^{\infty}\frac{\sin n|\theta|}{n} + F\sum_{n=1}^{\infty}\frac{\sin n|\theta|}{n}\right)$, where E and F are positive numbers such that $E \geqq \left|2\frac{\rho_\infty}{\rho_o}\frac{W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} + 1\right|$ and $F \geqq \left|2\frac{\rho_\infty}{\rho_o}\frac{W_{k,m}^{'(n)}(z_o)}{W_{k,m}^{(n)}(z_o)} - 1\right|$ for all large values of n.

Actually fewer terms occur in equation (170) than in the dominate series $\sum_{n=1}^{\infty}\frac{\sin n|\theta|}{n} = \frac{\pi}{2} - \frac{|\theta|}{2}$. Therefore equations (170) and consequently equation (164c) are convergent for $\sigma = 0$. Similarly, the convergence for $\sigma > 0$ can be shown.

$\underline{\theta_o = \frac{\pi}{2}}$.- This is the case which has been treated by Chaplygin and Lighthill. The condition $\theta = \theta_o$ occurs at the mouth of the aperture. Let $h/2$ be the half width of the mouth. Then,

$$Y = \frac{h}{2}$$

$$= \frac{\rho_o Q}{\pi \rho_\infty q_\infty}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right)\frac{\rho_\infty W'^{(n)}_{k,m}(z_o)}{\rho_o W^{(n)}_{k,m}(z_o)} + \right.$$

$$\left.\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\right]$$

$$= \frac{\rho_o Q}{\pi \rho_\infty q_\infty}\left[1 + 8\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2 - 1}\frac{\rho_\infty W'^{(n)}_{k,m}(z_o)}{\rho_o W^{(n)}_{k,m}(z_o)} + \right.$$

$$\left.\sum_{n'=2}^{\infty}\frac{(-1)^{n'-1}}{2n'-1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\right] \tag{171}$$

If, at $X = \infty$, the width of the jet is h_∞, then, by definition,

$$Q = \frac{\rho_\infty q_\infty}{\rho_o} h_\infty \tag{172}$$

Substituting this value into equation (171),

$$\frac{h}{h_\infty} = \frac{2}{\pi}\left[1 + 8\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2 - 1}\frac{\rho_\infty}{\rho_o}\frac{W'^{(n)}_{k,m}(z_o)}{W^{(n)}_{k,m}(z_o)} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{2n-1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\right] \tag{173}$$

Making use of the result of the well-known Leibniz series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n - 1} = \frac{\pi}{4}$, equation (173) can be written

$$\frac{h}{h_\infty} = \frac{2}{\pi} + \frac{16}{\pi} \frac{\rho_\infty}{\rho_o} \sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^2 - 1} \frac{W'^{(n)}_{k,m}\left(z_o\right)}{W^{(n)}_{k,m}\left(z_o\right)} + \frac{2}{\pi}\left(\frac{\pi}{4} - 1\right) + \frac{2}{\pi}\left(\frac{\pi}{4}\right)$$

$$= 1 + \frac{16}{\pi} \frac{\rho_\infty}{\rho_o} \sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^2 - 1} \frac{W'^{(n)}_{k,m}\left(z_o\right)}{W^{(n)}_{k,m}\left(z_o\right)} \tag{174}$$

Just as a check for the case of incompressible flow, the limiting case of $a^* \longrightarrow \infty$ or $\sigma \longrightarrow \infty$ may be considered.

First, considering equation (167) as σ becomes very large and n is finite,

$$\frac{W'^{(n)}_{k,m}(z)}{W^{(n)}_{k,m}(z)} = \overline{k} - \frac{1}{2} - \frac{m^2 - \left(k - \frac{1}{2}\right)^2}{z} \frac{e^{-\frac{z}{2}} z^{k-1}}{e^{-\frac{z}{2}} z^k}$$

$$= \left[\overline{k} - \frac{1}{2} - \overline{m}^2 + \left(\overline{k} - \frac{1}{2z}\right)^2\right] \tag{175}$$

The above equation tends to $-\frac{1}{2}$ when $\sigma \longrightarrow \infty$, owing to the fact that $\overline{k}$ and $\overline{m}$ both become zero as $\sigma \longrightarrow \infty$ $(z \longrightarrow \infty)$. Substituting the above equation into equation (174),

$$\frac{h}{h_\infty} = 1 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{4n^2 - 1}$$

$$= 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n - 1} + \frac{2}{\pi}\left[-\sum_{n'=1}^{\infty}\frac{(-1)^{n'-1}}{2n' - 1} + 1\right]$$

$$= 1 + \frac{2}{\pi} \tag{176a}$$

Or, inversely,

$$\frac{h_\infty}{h} = \frac{\pi}{2 + \pi} = 0.6110 \tag{176b}$$

which checks with the well-known result of Kirchoff.

For the case $\sigma_0 = 0$ $\left(M_\infty = 1\right)$ the width ratio h_∞/h has been calculated from equation (174). It is found to be 0.746 by summing terms up to $n = 10$. It is expected that the error is of the order -0.002. Consequently, it checks reasonably well with the Lighthill result $\left(h_\infty/h = 0.7447\right)$ which is based on the exact Chaplygin function.

The Johns Hopkins University
Baltimore, Md., April 3, 1951

APPENDIX A

SYMBOLS

a^*	velocity of sound
a_o^*	stagnation velocity of sound
a,b,c,d	constants
A,B,C,D	constants
${}_1F_1(\alpha,\beta;\ z)$	confluent hypergeometric function
${}_2F_1(a,b,c;\ z)$	hypergeometric function
h	width of jet
h_∞	width of jet at $X = \infty$

$$K = \left(\frac{\rho_o}{\rho}\right)^2\left(1 - M^2\right)$$

$$\tilde{K}_1 = a\sigma$$

$$\tilde{K}_2 = \frac{a\sigma}{1 + c\sigma}$$

$$\tilde{K}_3 = \frac{a\sigma(1 + b\sigma)}{(1 + c\sigma)^2}$$

$$k = \frac{-n\lambda(c - 2b)}{2c^2}\sqrt{\frac{a}{b}} = \nu\left(\frac{2}{c} - \frac{1}{b}\right) \quad \text{if} \quad \frac{ab}{c^2} = 1$$

$$\bar{k} = \frac{k}{z}$$

M	local Mach number
M_∞	free-stream Mach number

$M_{k,m}^{(n)}(z)$ Kummer function

$$m^2 = \frac{1}{4} - \frac{n^2\lambda^2 a(c - b)}{c^4} = \frac{1}{4} - \frac{\nu^2}{c^2}\left(\frac{a}{c} - 1\right) \quad \text{if} \quad \frac{ab}{c^2} = 1$$

$$\overline{m} = \frac{m}{z}$$

n positive integer

p pressure

p_o stagnation pressure, pressure in vessel

p_∞ free-stream pressure, pressure surrounding jet

Q arbitrary function of q $(Q(q))$

Q rate of total discharge

q local velocity

u X component of velocity

v Y component of velocity

$W_{k,m}^{(n)}(z)$ Whittaker function

$$W_{k,m}^{'(n)}(z) = \frac{dW_{k,m}^{(n)}(z)}{dz}$$

X,Y coordinates in physical plane

Z complex variable $(X + iY)$

$$z = \frac{2n\lambda}{c^2}\sqrt{ab}(1 + c\sigma) = \frac{2n\lambda}{c}(1 + c\sigma) \quad \text{if} \quad \frac{ab}{c^2} = 1$$

$$z_o = \frac{2n\lambda}{c^2}\sqrt{ab}\left(1 + c\sigma_o\right)$$

α, β constants

$\beta = \dfrac{1}{\gamma - 1}$

$\gamma = C_p/C_v = 1.4$

$\epsilon = Q(q)$ $\left(\text{defined by } \epsilon\left(\dfrac{d\epsilon}{dq}\right)^2 = -\dfrac{1 - M^2}{q^2}\right)$

$\zeta = \left(\dfrac{3}{2}\right)^2 \dfrac{\theta^2}{\epsilon^3} = \dfrac{\theta^2}{\Omega^2}$ (supersonic)

$\zeta = \left(\dfrac{3}{2}\right)^2 \dfrac{\theta^2}{\epsilon^3} = \dfrac{\theta^2}{\omega^2}$ (subsonic)

θ inclination of velocity vector

θ_o coordinate of source location, hodograph plane

θ_o slope of nozzle wall

λ characteristic coordinate $(\overline{\Omega} + \theta)$

$\lambda = \dfrac{\pi}{\theta_o}$

λ_o, μ_o coordinates of source location in hodograph plane

μ characteristic coordinate $(-\overline{\Omega} + \theta)$

$\mu_1 = \dfrac{\gamma + 1}{\gamma - 1}$

ν $\displaystyle\int_q^{a^*} \frac{\rho_o}{\rho} \frac{1 - M^2}{q} dq$

ν positive constant dependent on boundary conditions $(n\lambda)$

ρ local density

ρ_o stagnation density

ρ_∞ free-stream density

$$\sigma = \int_q^{a^*} \frac{\rho_o}{\rho} \frac{dq}{q}$$

$$\tau = \frac{\gamma - 1}{2}\left(\frac{q}{a_o^*}\right)^2$$

ϕ potential function

ψ stream function

$$\Omega = \int_{a^*}^{q} \frac{\left(M^2 - 1\right)^{1/2}}{q} dq \qquad \text{(supersonic)}$$

$$\overline{\Omega} = \Omega + \frac{a'}{b'}$$

$$\omega = \int_q^{a^*} \frac{\left(1 - M^2\right)^{1/2}}{q} dq \qquad \text{(subsonic)}$$

ω_o coordinate of source location in hodograph plane

ω_∞ ω in free stream

APPENDIX B

PROPERTIES OF WHITTAKER FUNCTIONS

Here are presented a few important properties of the Whittaker function which are useful in the present investigation. Some of the properties are given by Sharma in reference 59. He shows that:

Theorem I:

> The functions $W_{k,m}(z)$ and $W_{k,m+1}(z)$ cannot have a common zero (root). All roots of $W_{k,m}(z)$ are simple; $\overline{W}_{k,m}(z)$ and $W_{k-1,m}(z)$ cannot have a common root. Between any two consecutive zeros of $W_{k,m}(z)$ lies one and only one zero of $W_{k-1,m}(z)$.

The proof is rather simple. See the reference.

Next, consider the Whittaker equation:

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} - \frac{m^2 - \frac{1}{4}}{z^2}\right)W = 0 \tag{B1}$$

where $W = W_{k,m}^{(n)}(z)$, the Whittaker function. In the present case,

$$z = 2n\lambda\left(\frac{ab}{c^2}\right)^{1/2}\left(\frac{1}{c} + \sigma\right) > 0 \qquad \left(a > c > b > 0\right), \frac{ab}{c^2} = 1 \tag{B2}$$

where n is a positive integer and λ is a positive constant.

$$k = n\lambda\left(\frac{2}{c} - \frac{1}{b}\right) < 0 \tag{B3}$$

$$\left.\begin{aligned} m^2 &= \frac{1}{4} - \frac{n^2\lambda^2}{c^2}\left(\frac{a}{c} - 1\right) < 0 \quad \text{for} \quad n^2\lambda^2 > \frac{c^3}{4(a - c)} \\ m^2 &= \frac{1}{4} - \frac{n^2\lambda^2}{c^2}\left(\frac{a}{c} - 1\right) > 0 \quad \text{for} \quad n^2\lambda^2 < \frac{c^3}{4(a - c)} \end{aligned}\right\} \tag{B4}$$

In this case $m^2 - \frac{1}{4} < 0$; the powerful Wintner theorem (reference 74) which requires $k < 0$ and $m^2 - \frac{1}{4} > 0$ to make $(-1)^n \frac{d^n W_{k,m}(z)}{dz^n} \geqq 0$ $(0 \leqq z \leqq \infty)$ cannot be applied. Actually, $W_{k,m}^{(n)}(z)$ may oscillate in the range $-\frac{1}{c} < \sigma < 0$. But the monotonic positive nature of $W_{k,m}^{(n)}(z)$ for $\sigma \geqq 0$ can be proved as follows:

Theorem II:

For $n\lambda > 0$ and $0 \leqq \sigma < \infty$ $(1 \geqq M > 0)$, then $W_{k,m}^{(n)}(z) > 0$.

The proof is as follows:

If equation (B1) is multiplied by $\left(\frac{dz}{d\sigma}\right)^2 = 4n^2\lambda^2\left(\frac{ab}{c^2}\right)$, there can be written

$$\frac{d^2W}{d\sigma^2} - \frac{n^2\lambda^2 a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} W = 0$$

or, more properly,

$$\frac{d^2W}{d\sigma^2} = \frac{n^2\lambda^2 a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} W \tag{B5}$$

By the given conditions, as long as $\sigma \geqq 0$,

$$\frac{n^2\lambda^2 a\sigma(1 + b\sigma)}{(1 + c\sigma)^2} \geqq 0 \tag{B6}$$

Therefore $\frac{d^2W}{d\sigma^2}$ and W have the same sign. But $W = W_{k,m}^{(n)}(z)$ is asymptotic to $e^{-\frac{z}{2}} z^k = e^{-n\lambda\left(\frac{1}{c} + \sigma\right)} \left[2n\lambda\left(\frac{1}{c} + \sigma\right)\right]^{n\lambda\left(-\frac{2}{c} + \frac{1}{b}\right)} > 0$ for large values of σ and tends to 0 as $\sigma \longrightarrow 0$. Hence $W_{k,m}^{(n)}(z)$ curves downward as $\sigma \longrightarrow \infty$ and so must continue to do so while $W^{-1} \frac{d^2W}{d\sigma^2} > 0$. This proves the theorem.

There is another interesting feature of the Whittaker equation. As long as m^2 is real and the function representing the boundary condition is real, $W_{k,m}^{(n)}(z)$ is always real, no matter whether m is imaginary or not. For the present investigation, m is imaginary, but $W_{k,m}^{(n)}(z)$ is real $(0 < z < \infty)$. This question has puzzled the author for some time.

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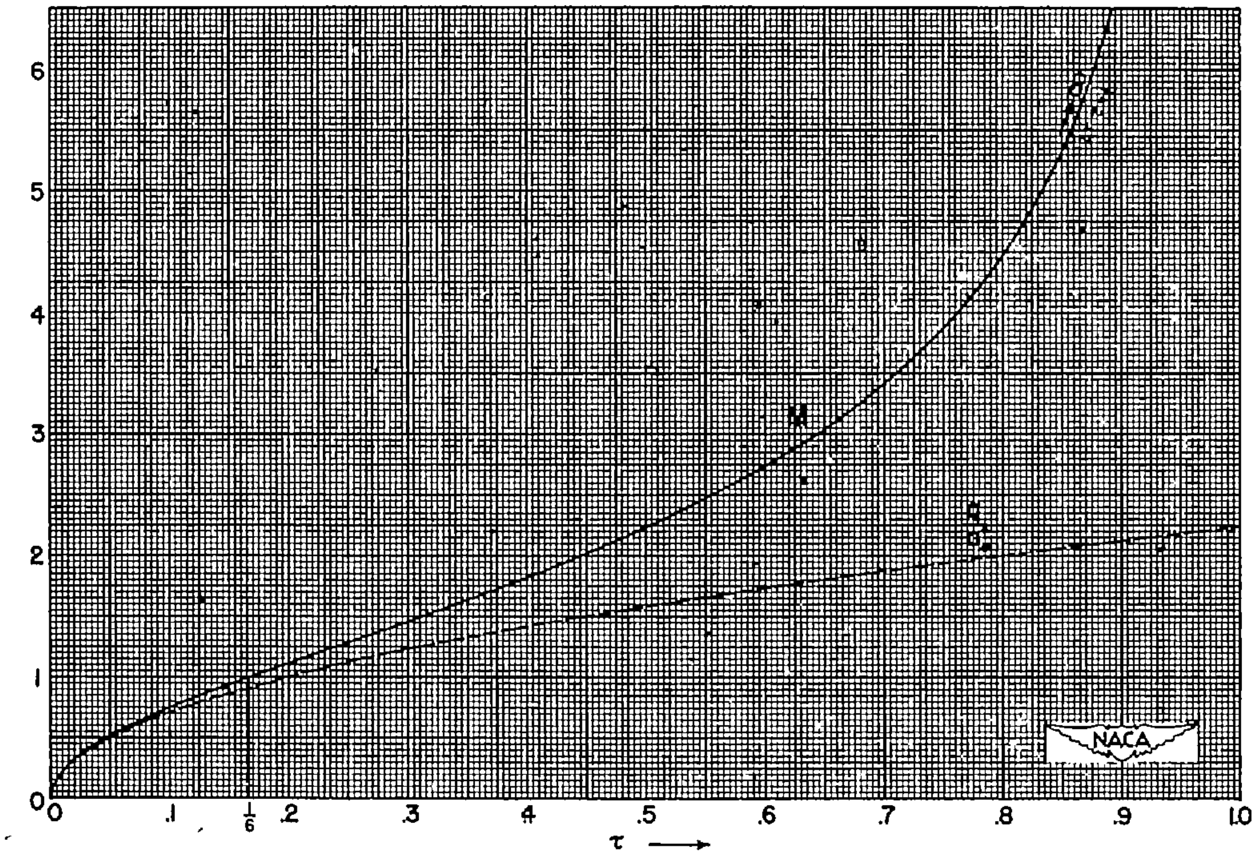


Figure 1.- Plot of $\frac{q}{a_o^*}$ and Mach number M as a function of τ. $\gamma = 1.4$.

$M = \left(\frac{2\beta\tau}{1 - \tau}\right)^{1/2}$; $\frac{q}{a_o^*} = \left(\frac{2\tau}{\gamma - 1}\right)^{1/2}$. $M = 1$, $\frac{q}{a_o^*} = 0.9192$ at $\tau = 1/6$;

$M = \sqrt{2}$, $\frac{q}{a_o^*} = 1.200$ at $\tau = 0.286$; $M = \sqrt{5}$, $\frac{q}{a_o^*} = 1.583$ at $\tau = 0.5$;

$M = \infty$, $\frac{q}{a_o^*} = \sqrt{5}$ at $\tau = 1.00$.

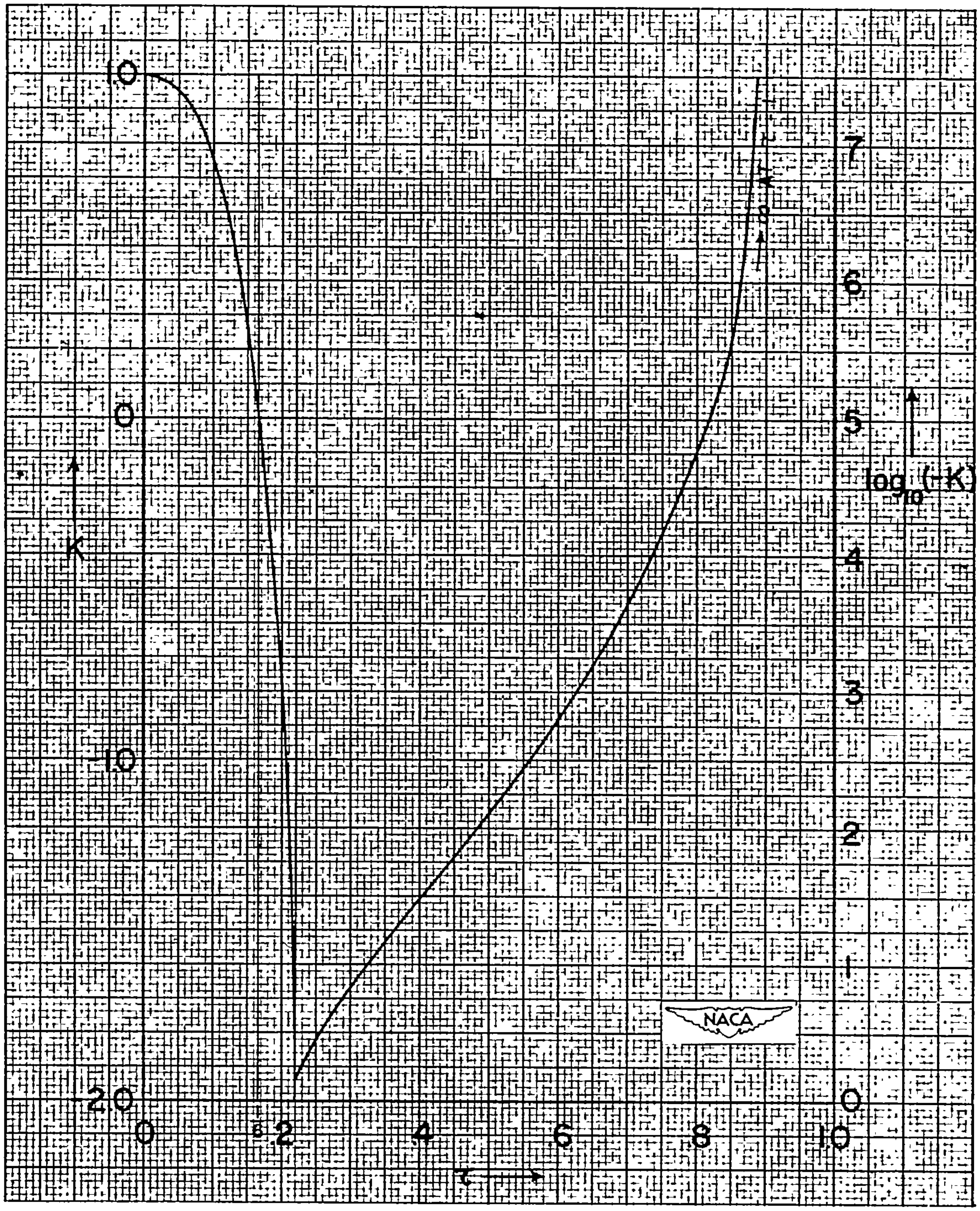


Figure 2.- Plot of K as a function of τ for $0 < \tau < 0.22$ and $\log(-K)$ as a function of τ for $0.22 < \tau < 1.0$. $\mu_1 = 6$ for $\gamma = 1.4$. $K = \left(\frac{\rho_0}{\rho}\right)^2(1 - M^2) = \frac{1 - \mu_1\tau}{(1 - \tau)^{\mu_1}}$. $M = 0$, $K = 1$, $\tau = 0$; $M = 1$, $K = 0$, $\tau = 1/6$; $M = \infty$, $K = -\infty$, $\tau = 1$.

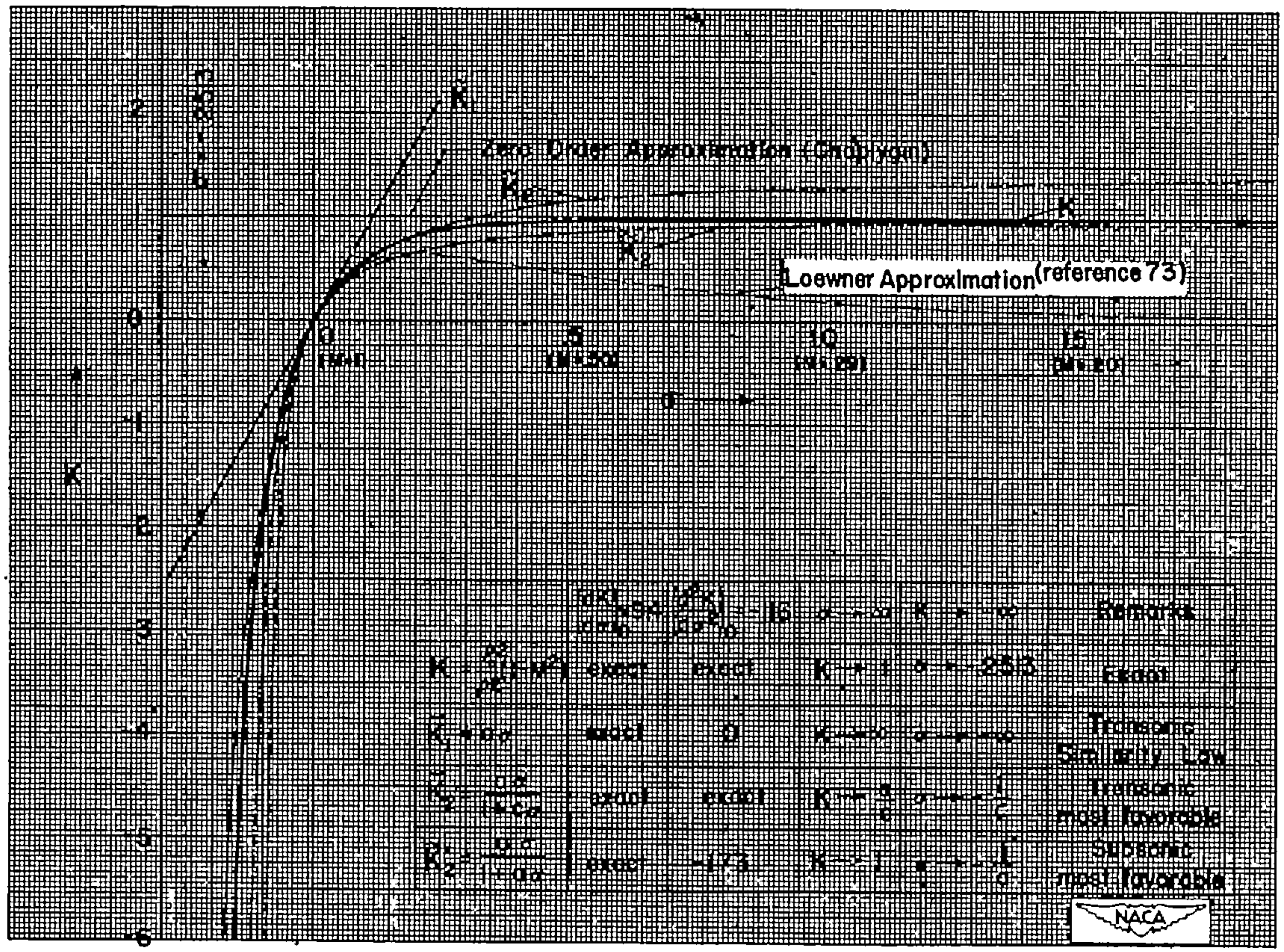


Figure 3.- Plot of K as a function of σ. Horizontal asymptote: $K = 1$; vertical asymptote: $\sigma = -0.2513$. $M = 0$, $K = 1$, $\sigma = \infty$; $M = 1$, $K = 0$, $\sigma = 0$; $M = \infty$, $K = -\infty$, $\sigma = -0.2513$.

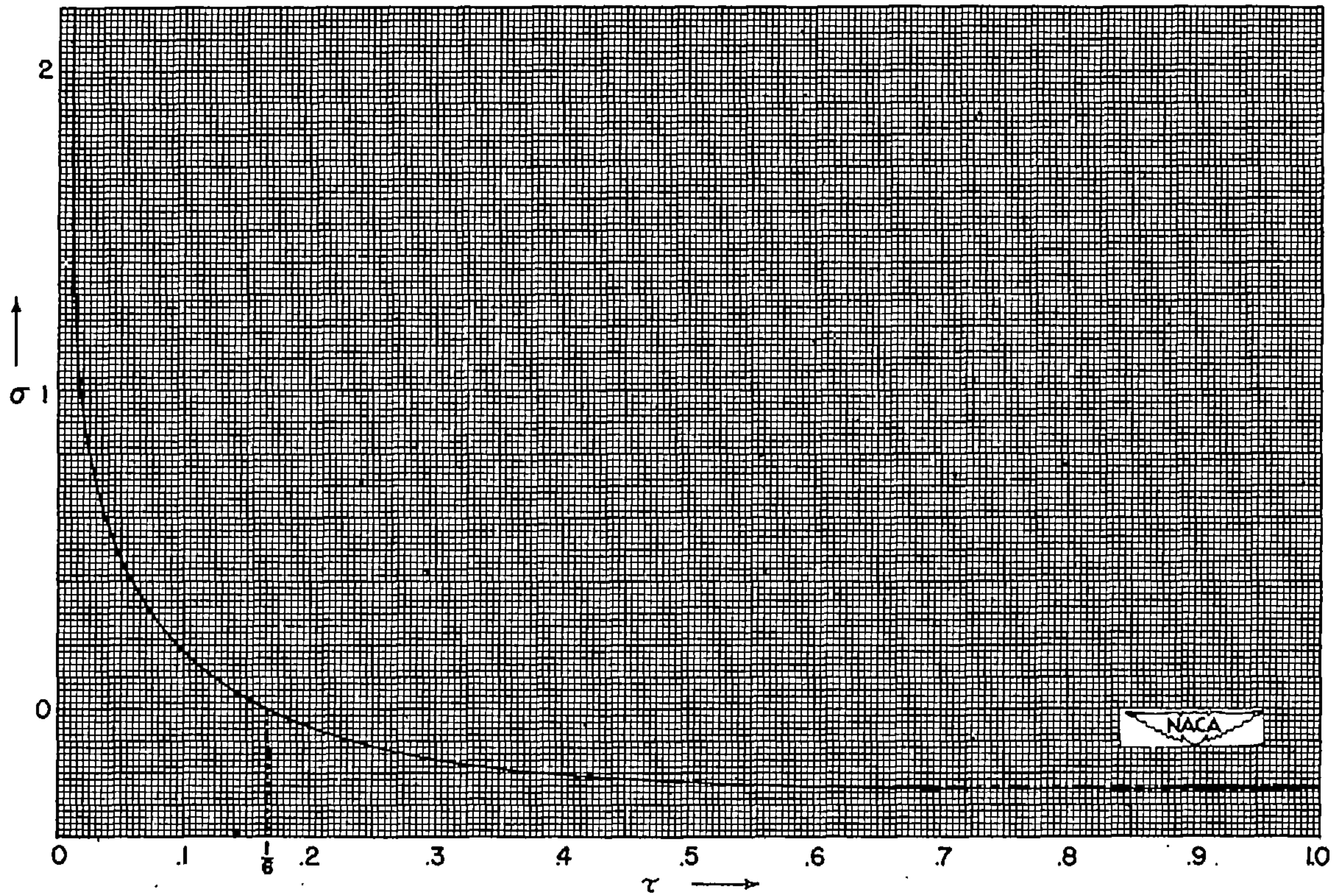
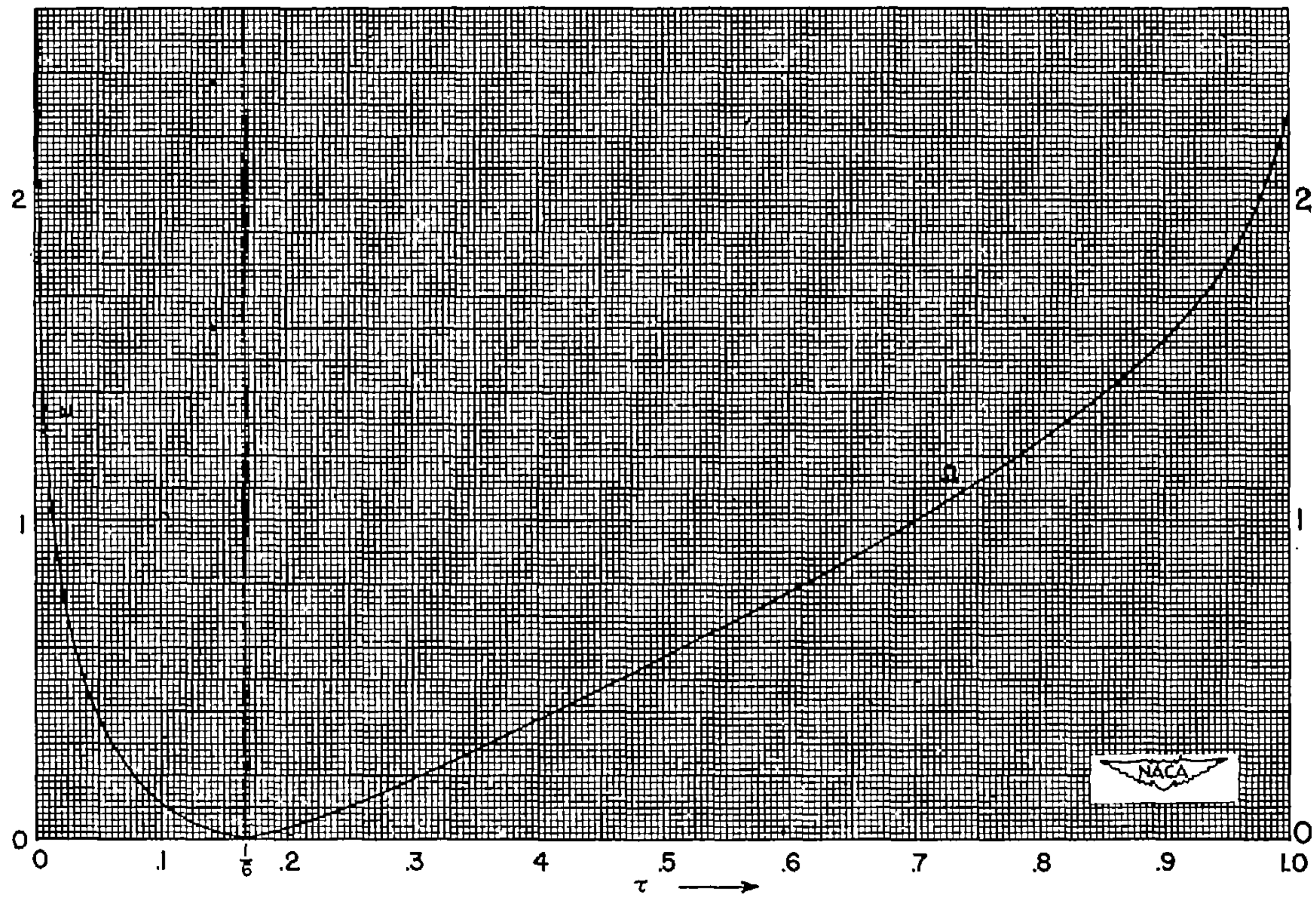
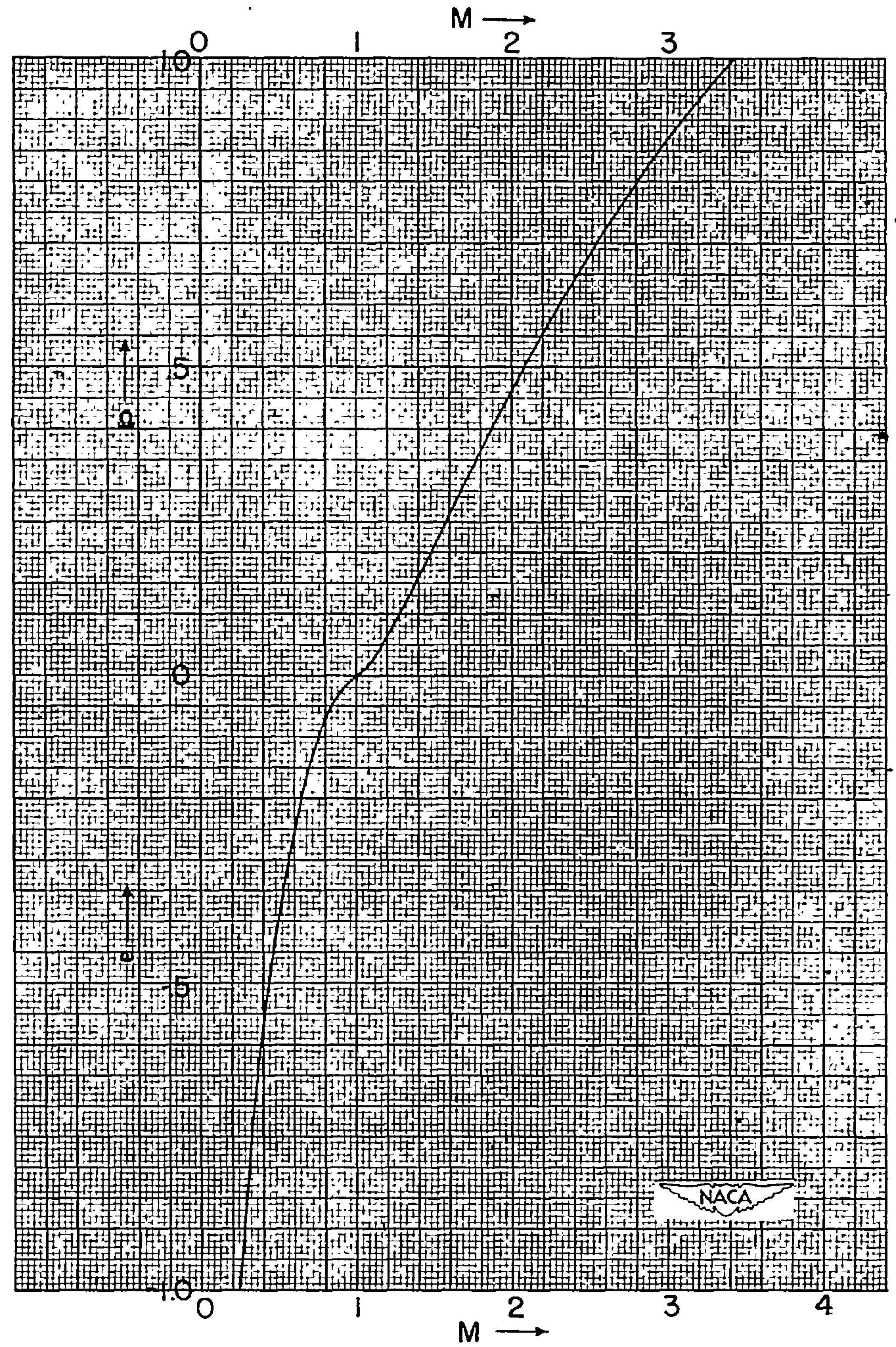


Figure 4.- Plot of σ as a function of τ. $\sigma = -0.251251 - (1 - \tau)^{\frac{1}{2}}\left[1 + \frac{1 - \tau}{3} + \frac{(1 - \tau)^2}{5}\right] + \tanh^{-1}(1 - \tau)^{\frac{1}{2}}$.
$M = 0$, $\tau = 0$, $\sigma = \infty$; $M = 1$, $\tau = 1/6$, $\sigma = 0$; $M = \infty$, $\tau = 1.0$, $\sigma = -0.2513$.



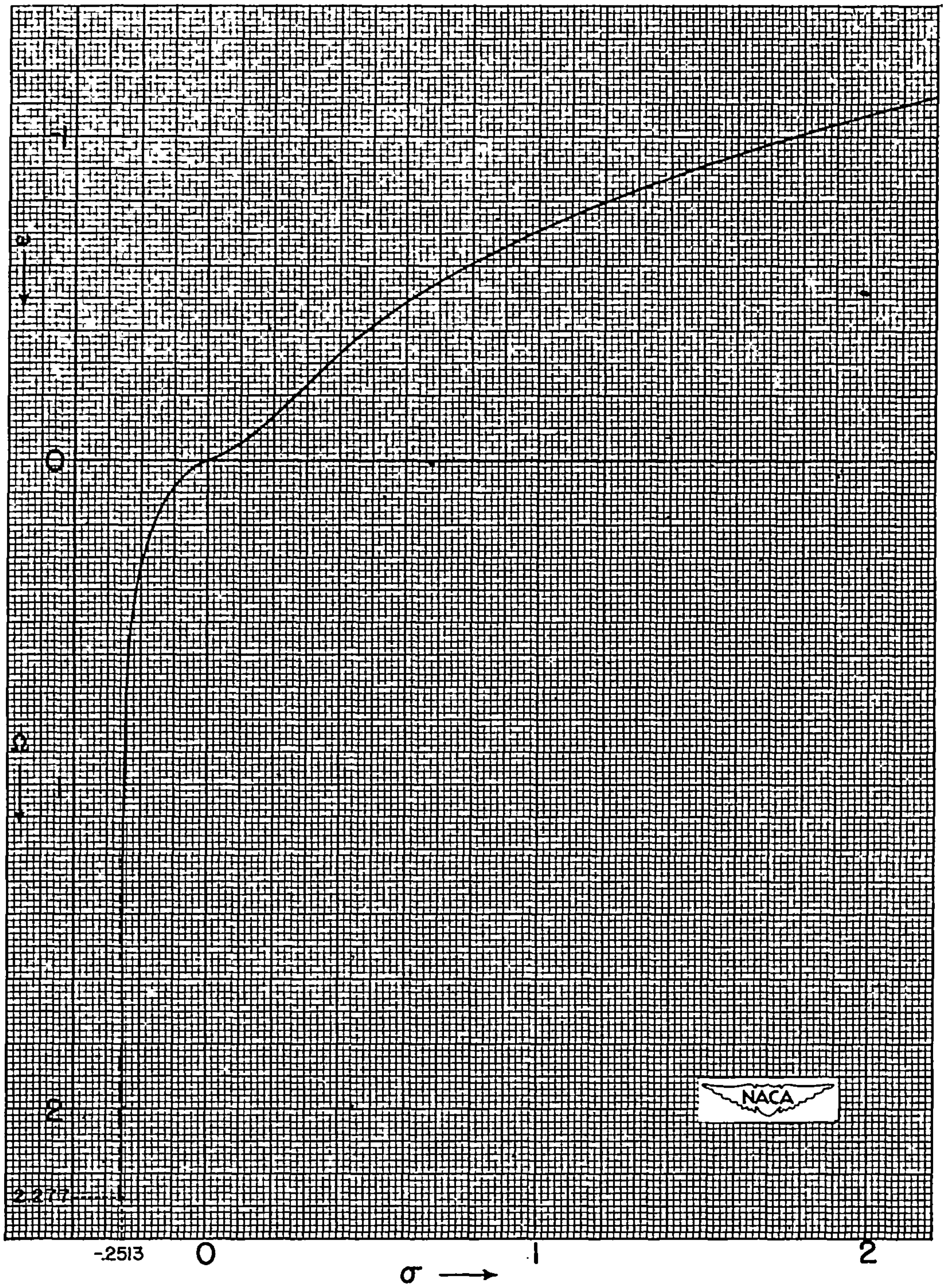
(a) As functions of τ. See equations (18c) and (22c). $M = 0$, $\omega = -\infty$, at $\tau = 0$; $M = 1$, $\omega = 0$, at $\tau = 1/6$; $M = 1$, $\Omega = 0$, at $\tau = 1/6$; $M = \infty$, $\Omega = 2.277$, at $\tau = 1.0$.

Figure 5.- Plots of ω and Ω against various functions.



(b) As functions of M. See equations (18c) and (22c). M = 0, $\omega = -\infty$; M = 1, $\omega = 0$; M = 1, $\Omega = 0$; M = ∞, $\Omega = 2.277$.

Figure 5.- Continued.



(c) As functions of σ. $M = 0$, $\sigma = \infty$, $\omega = -\infty$; $M = 1$, $\sigma = 0$, $\omega = 0$; $M = 1$, $\sigma = 0$, $\Omega = 0$; $M = \infty$, $\sigma = 0.2513$, $\Omega = 2.277$.

Figure 5.- Concluded.

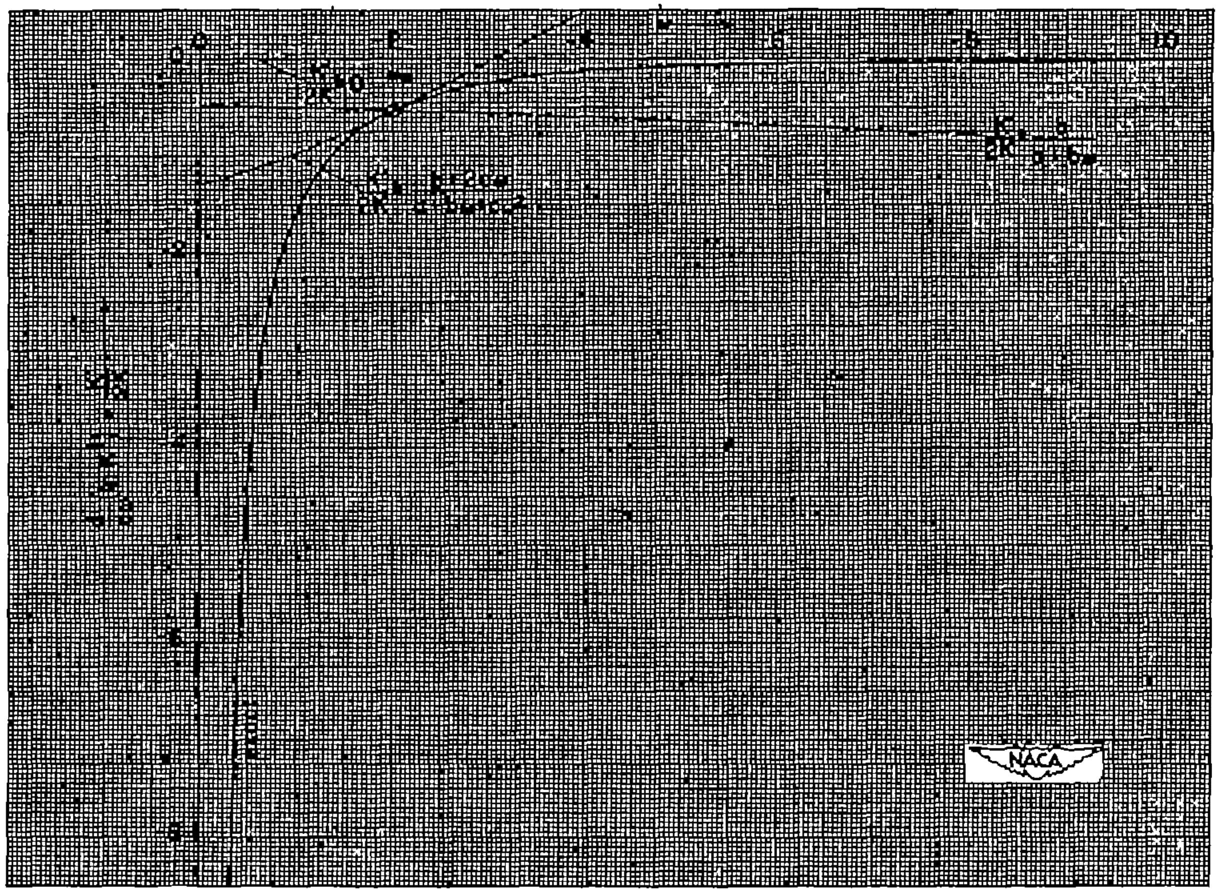


Figure 6.- Plot of $\dfrac{d\left(\log_e K^{1/2}\right)}{d\omega}$ as a function of ω. Horizontal asymptote: $\dfrac{K'}{2K} = 0$; vertical asymptote: $\omega = 0$. $M = 0$, $\dfrac{d\left(\log_e K^{1/2}\right)}{d\omega} = 0$, $\omega = \infty$; $M = 1$, $\dfrac{d\left(\log_e K^{1/2}\right)}{d\omega} = -\infty$, $\omega = 0$.

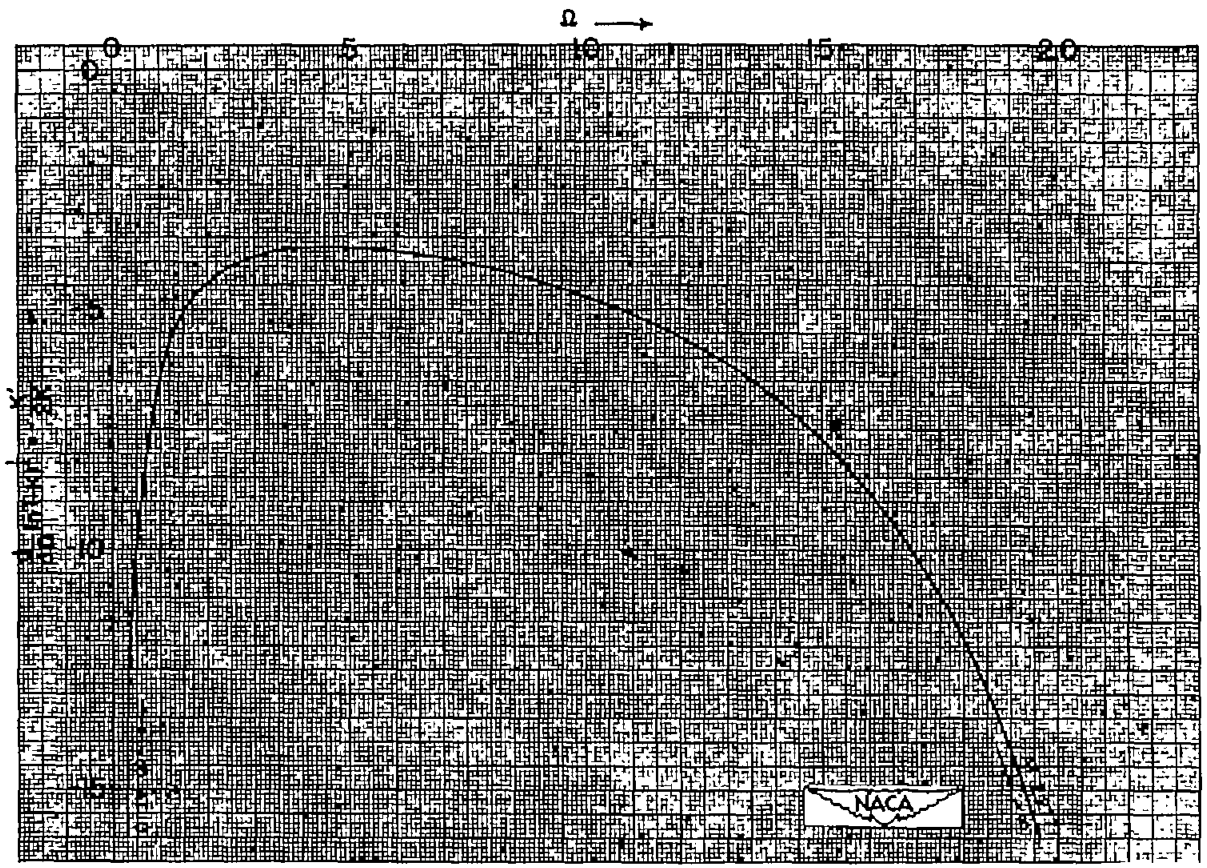


Figure 7.- Plot of $\dfrac{d\left[\log_e(-K)^{1/2}\right]}{d\Omega}$ as a function of Ω. Vertical asymptotes: $\Omega = 0$, $\Omega = 2.277$. $M = 1$, $\dfrac{d\left[\log_e(-K)^{1/2}\right]}{d\Omega} = -\infty$, $\Omega = 0$; $M = \infty$, $\dfrac{d\left[\log_e(-K)^{1/2}\right]}{d\Omega} = -\infty$, $\Omega = 2.277$.

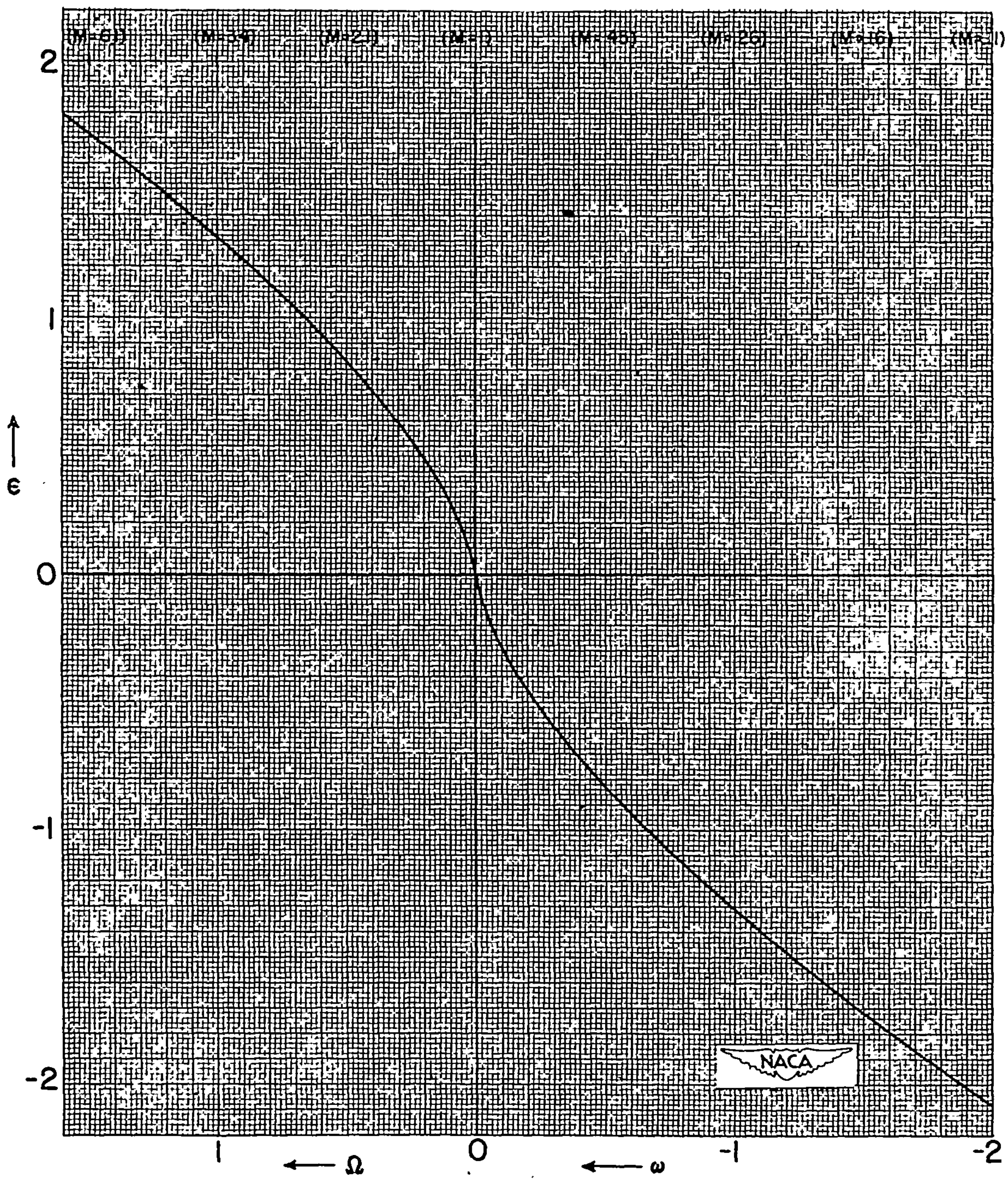


Figure 8.- Plot of ϵ as a function of ω and Ω. $M = 0$, $\omega = -\infty$, $\epsilon = -\infty$; $M = 1$, $\omega = 0$, $\epsilon = 0$; $M = 1$, $\Omega = 0$, $\epsilon = 0$; $M = \infty$, $\Omega = 2.277$, $\epsilon = 2.268$.

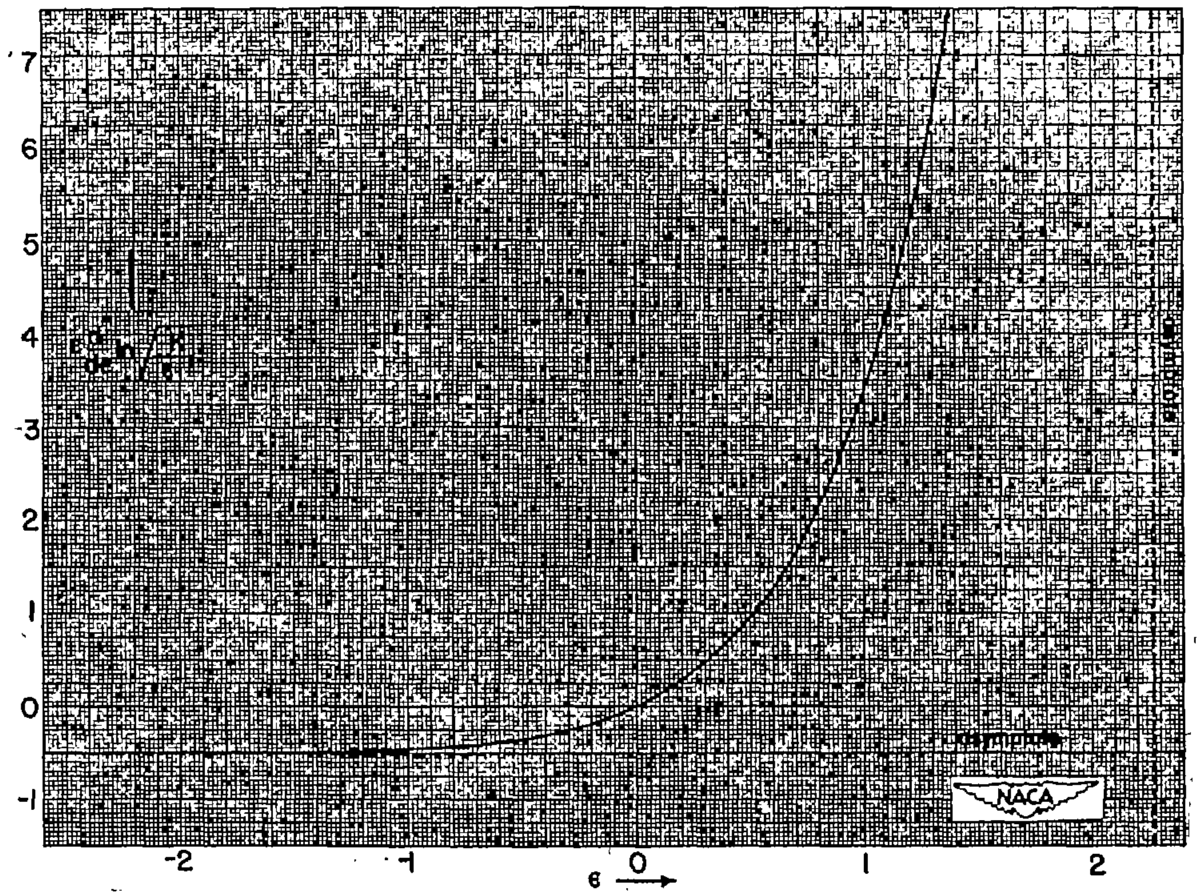


Figure 9.- Plot of $\epsilon \frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)^{1/2}\right]$ as a function of ϵ. $M = 0$, $\epsilon \frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)^{1/2}\right] = -1/2$, $\epsilon = -\infty$; $M = 1$, $\epsilon \frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)^{1/2}\right] = 0$, $\epsilon = 0$; $M = \infty$, $\epsilon \frac{d}{d\epsilon}\left[\log_e\left(\frac{-K}{\epsilon}\right)^{1/2}\right] = \infty$, $\epsilon = 2.268$.

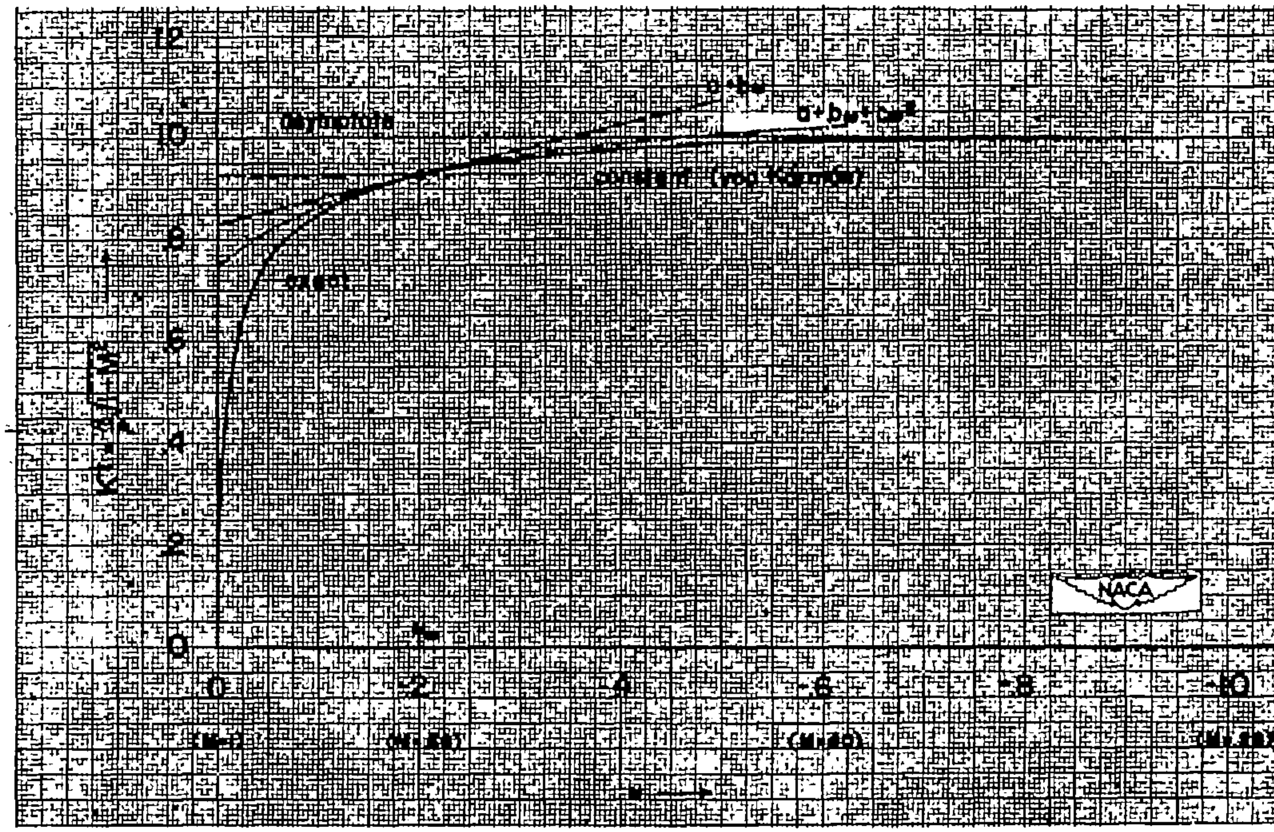


Figure 10.- Different order approximations to curve $K^{1/2} = \frac{\rho_0}{\rho}(1 - M^2)^{1/2}$ against ω at free-stream ω_∞. Horizontal asymptote: $K^{1/2} = 1$. $\tilde{K}_0{}^{1/2} = \frac{\rho_0}{\rho_\infty}(1 - M_\infty{}^2)^{1/2}$, zero-order approximation by Von Kármán; $\tilde{K}_1{}^{1/2} = a + b\omega$, first-order approximation; $\tilde{K}_2{}^{1/2} = a + b\omega + c\omega^2$, second-order approximation.

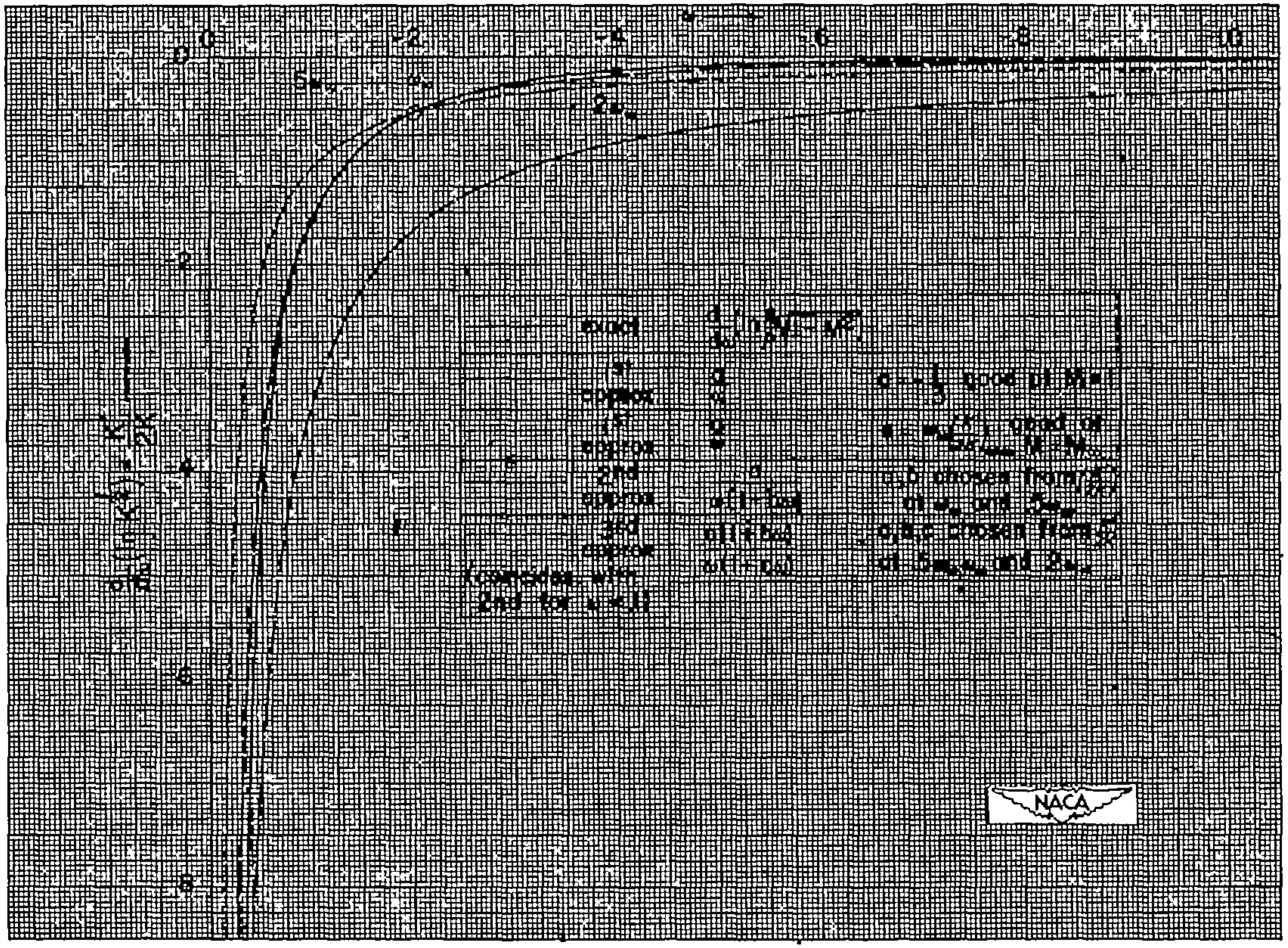


Figure 11.- Other subsonic approximations in comparison with $\frac{d}{d\omega}\left(\log_e K^{1/2}\right)$ in figure 6. Horizontal asymptote: $\frac{K'}{2K} = 0$; vertical asymptote: $\omega = 0$.

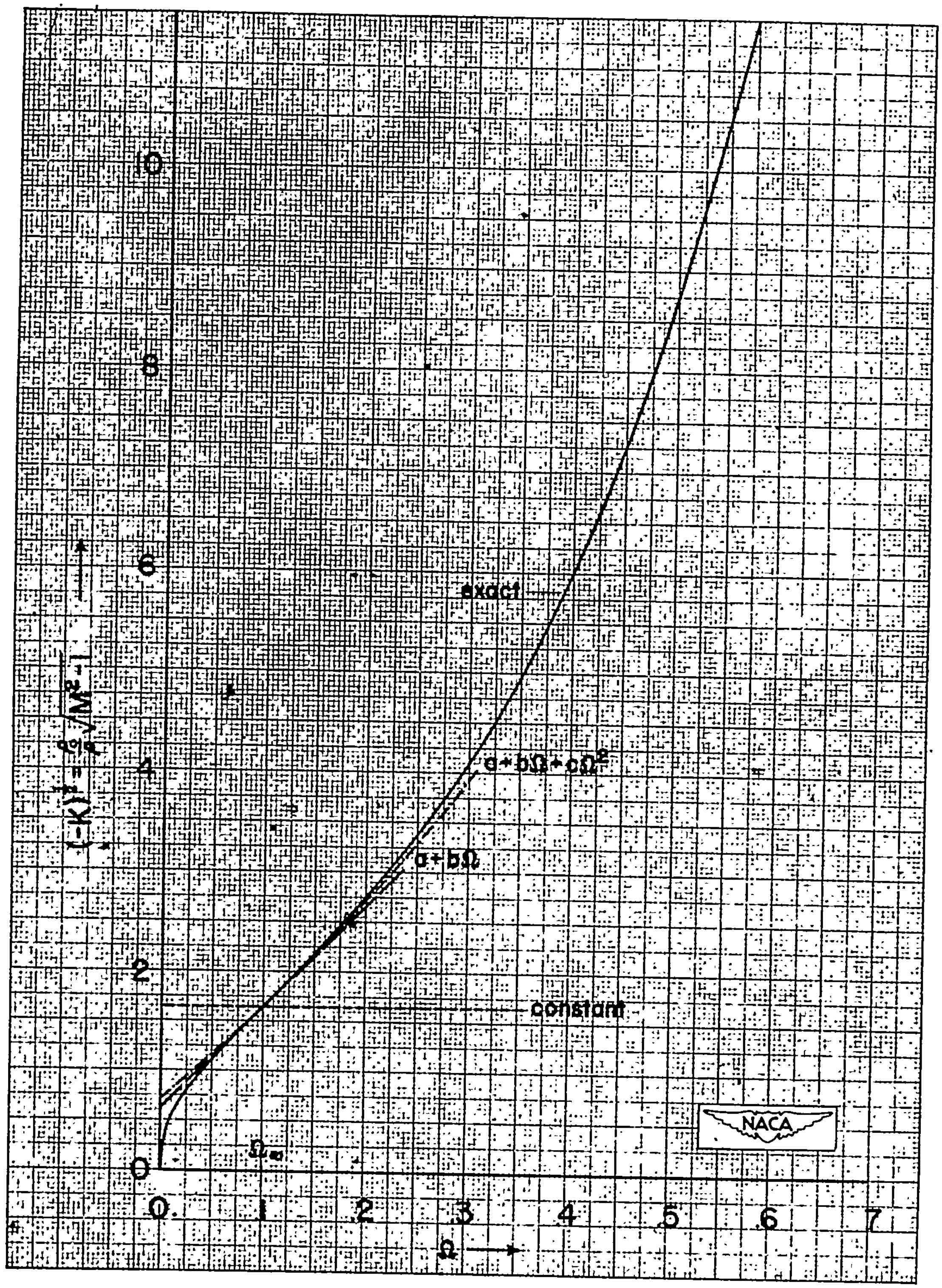


Figure 12.- Supersonic approximations in comparison with exact $\frac{\rho_o}{\rho}\sqrt{M^2 - 1}$ against Ω. Vertical asymptote: $\Omega = 2.277$.

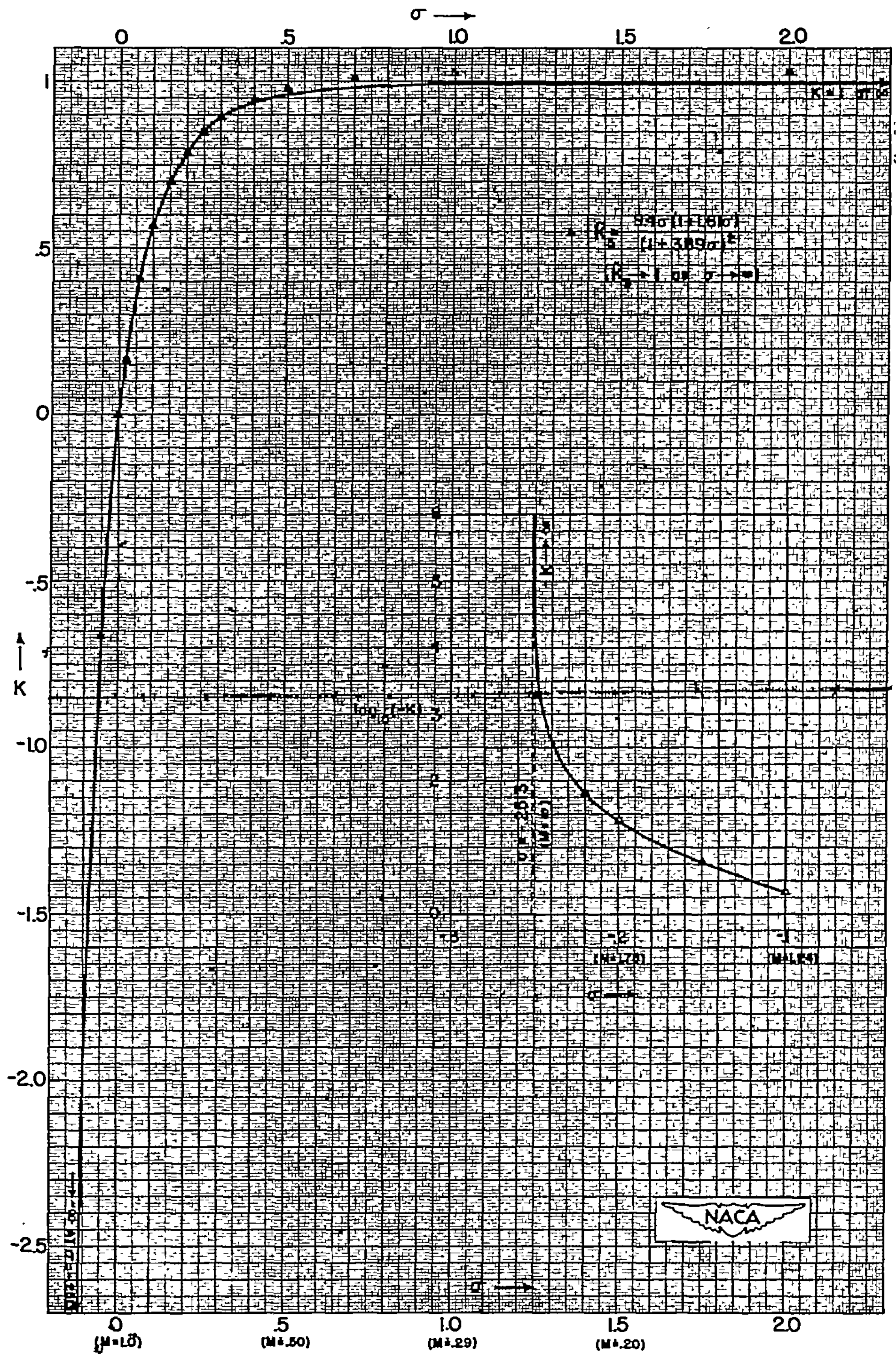


Figure 13.- Detailed comparison of third-order approximation with exact $K(\sigma)$. Solid line, exact; triangle, approximation. Horizontal asymptote: $K = 1$; vertical asymptote: $\sigma = -0.2513$. $a = 9.4$, $b = 1.61$, $c = 3.89$. $\frac{ab}{c^2} = 1$.

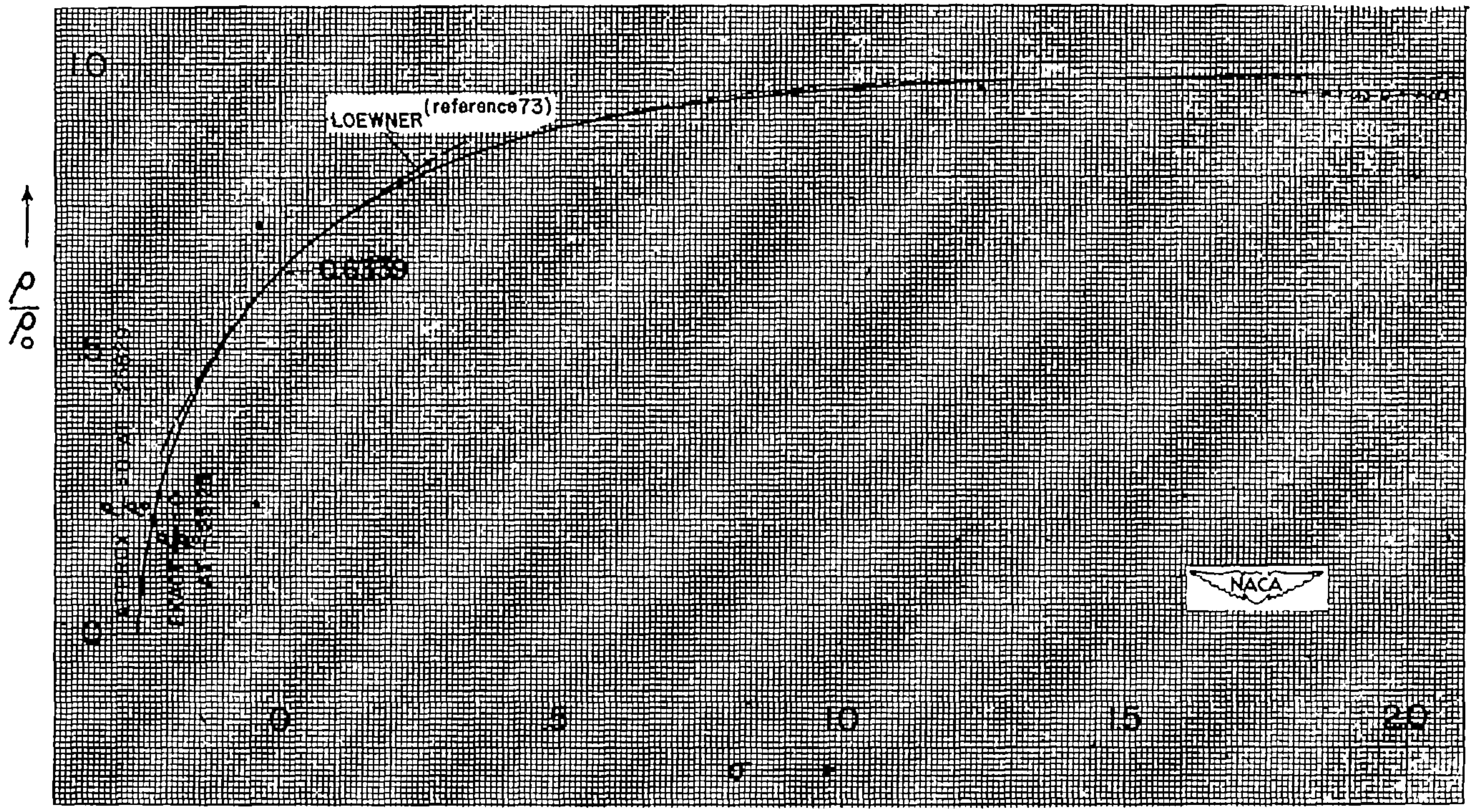


Figure 14.- Comparison of ρ_o/ρ in hypothetical gas law with exact gas law as a function of σ. Dotted line, exact; solid line, approximation; long dashed line, Loewner approximation (reference 73). Horizontal asymptote: $\frac{\rho}{\rho_o} = 1$. $\frac{\rho}{\rho_o} = 0.6339$ at $\sigma = 0$ $(M = 1)$.

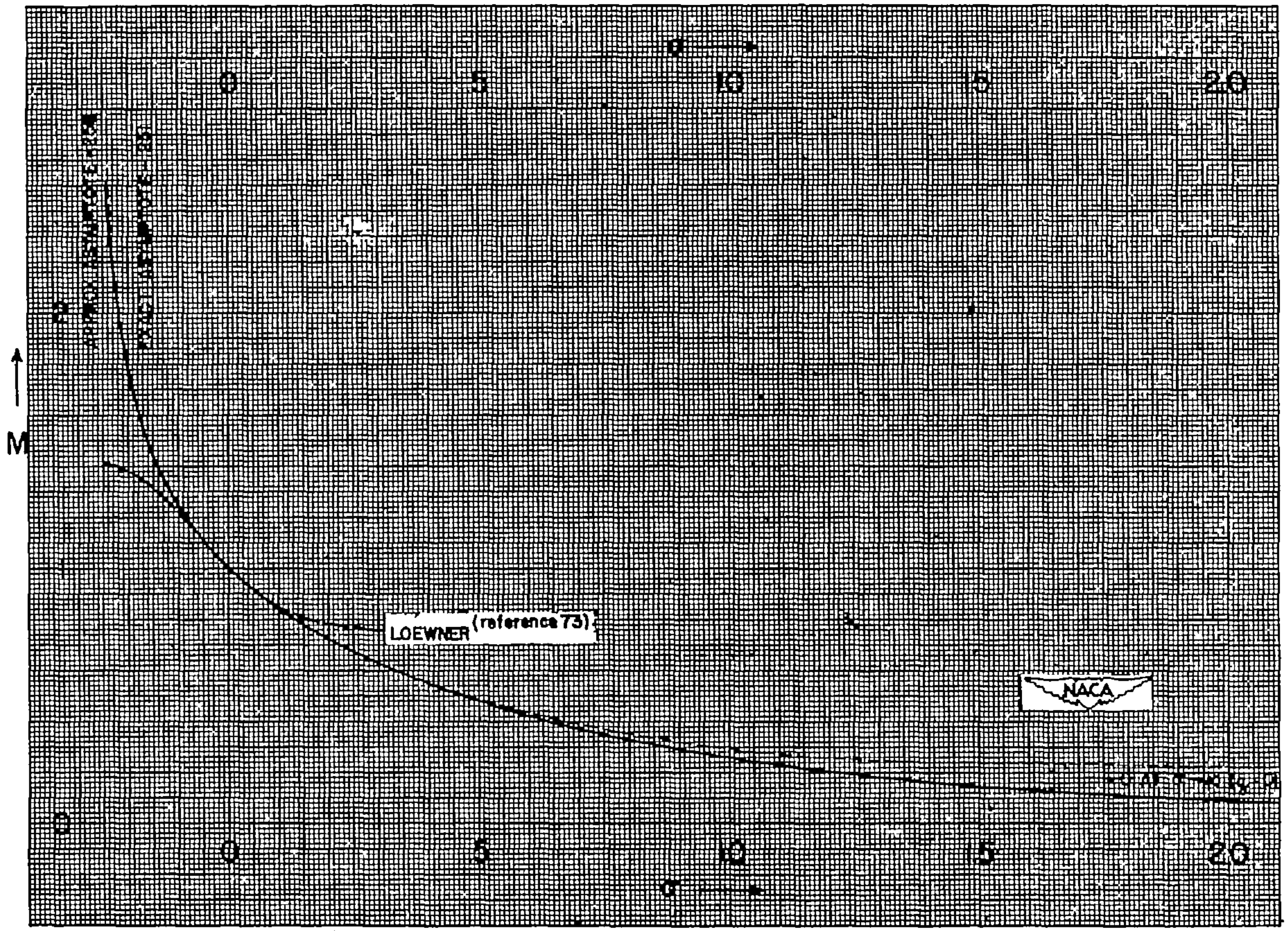
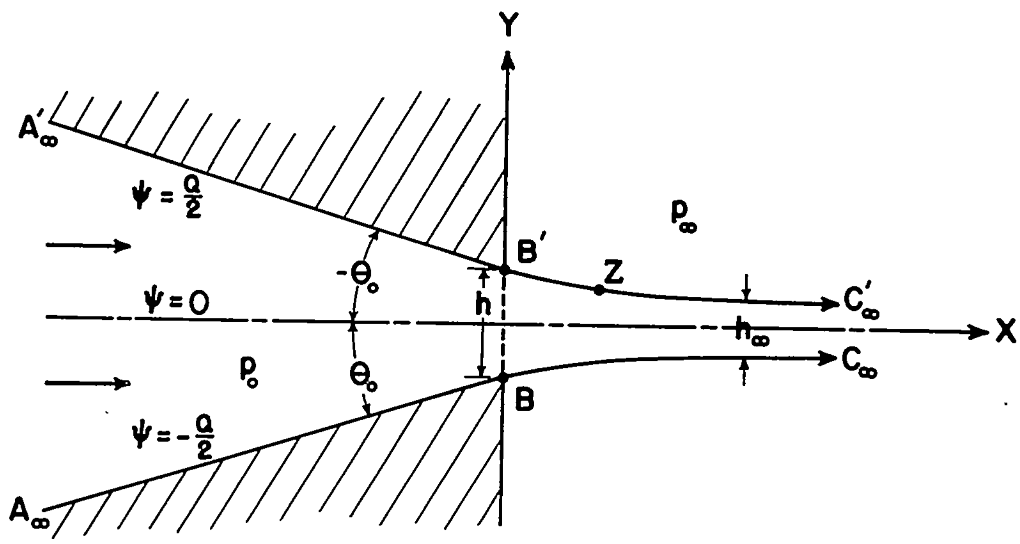
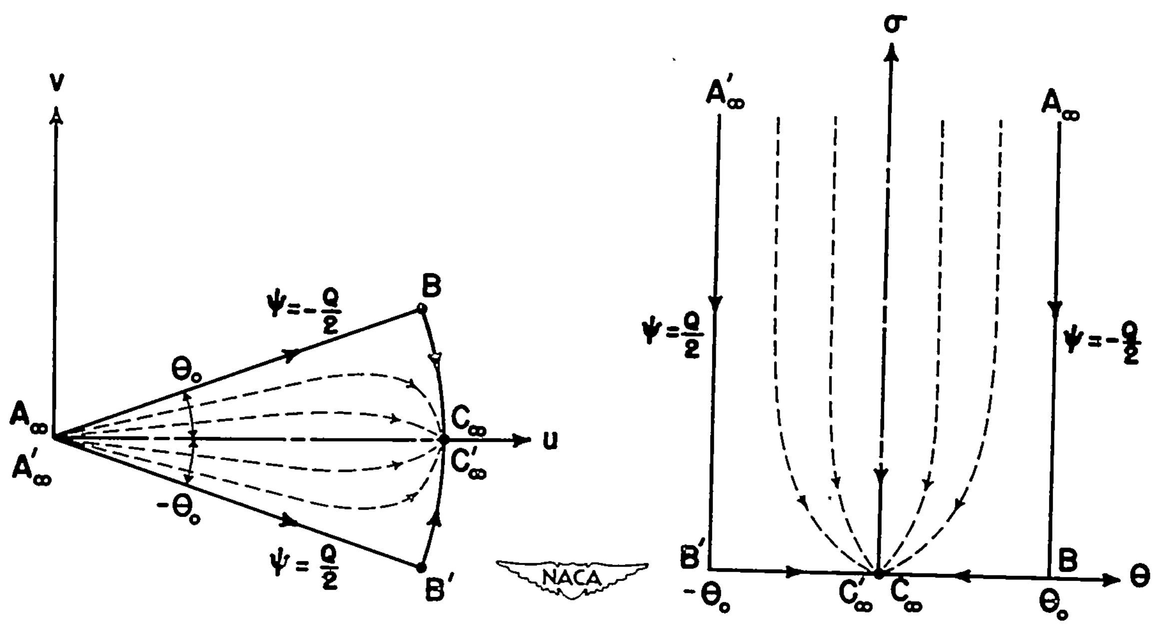


Figure 15.- Comparison of Mach number of hypothetical gas and exact gas.



(a) Physical plane.



(b) Hodograph plane ($q - \theta$).

(c) Hodograph plane ($\sigma - \theta$). $\gamma = 1.4$.
$\sigma_o = 0$; $\frac{p_\infty}{p_o} = 0.5283$; $\frac{\rho_\infty}{\rho_o} = 0.6339$

Figure 16.- Sketch of flow through an inclined-walled straight-edged aperture.

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